

# Bell inequalities and quantum mechanics

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We employ an arrangement of polarization analyzer loops to derive several simple Bell inequalities and then discuss the violation of one of them in light of quantum and classical interpretations of the data recorded. © 2002 American Association of Physics Teachers.  
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## I. INTRODUCTION

Inequalities of the Bell type<sup>1</sup> by themselves have nothing to do with quantum theory. Contexts as different as downhill skiers<sup>2</sup> and laundered socks<sup>3</sup> have been used by Meystre and Bell to demonstrate this. Although Bell inequalities are almost tautological expressions, they have attracted much attention because they allow one to see the experimental consequences of alternative views of physical reality,<sup>4</sup> which are conveniently labeled classical and quantum mechanical. The first experiment specifically designed to distinguish between opposing views was carried out by Freedman and Clauser in 1972,<sup>5</sup> and subsequent experimental efforts<sup>6</sup> have supported the first results, which are not compatible with the standard classical view that all natural world phenomena are local and realistic.

An insistence on both locality and realism is associated with the views expressed in the well-known paper by Einstein, Podolsky, and Rosen.<sup>7</sup> Because quantum theory contains elements that conflict with a local and realistic view, Einstein expressed his opinion that quantum theory must therefore be incomplete, implying the existence of a deeper theory based on still unrecognized “hidden” variables.

It is interesting to students of quantum mechanics at many levels of expertise to reflect on the impact of Bell inequality experiments on their view of the “real” world of nature. In my experience this impact is certainly felt even by first-year college students.

Here I will describe a thought experiment that uses an arrangement of optical analyzer loops, devices discussed in detail by French and Taylor in their excellent treatment of photon polarization.<sup>8</sup> The experimental setup allows an easy derivation of several Bell inequalities. The outcome of the experiment, however, violates these inequalities in an obvious way. These results lead to a discussion with a relatively narrow focus on reality, although it could easily be changed slightly to permit a larger discussion including locality as well.

## II. ANALYZER LOOPS AND THE CLAUSER EXPERIMENT

An analyzer loop is sketched in Fig. 1. It is a pair of birefringent crystals arranged with an air gap between them, and cut and positioned in such a way that a light beam entering the first crystal is divided into orthogonally polarized components that travel separately across the air gap and are then recombined into the original beam by the second crystal. Such a device is a kind of interferometer and not casually assembled, but an analyzer loop can be seen in operation in an excellent instructional movie entitled, “Interference in

Photon Polarization.”<sup>9</sup> Something very close to such a device is shown in Fig. 5-5 of Vol. III of the Feynman Lectures.<sup>10</sup>

What I will call a Clauser experiment, a close conceptual analog of the one carried out by Freedman and Clauser,<sup>5</sup> is set up using five analyzer loops, two to the left and three to the right of a photon source, as shown in Fig. 2. The thought experiment takes place in three stages, in each of which the right detector’s photon counts are recorded. The right recorder is triggered by the arrival of the left-going photon at its detector. Either detector can be located at the end of a long delay line if needed. The first crystals on either side select  $x$  and  $y$  polarizations, and the other crystals are oriented to analyze the polarization as labeled. As for notation, a bar is used to denote the orthogonal complement. For example,  $\bar{\theta} = \theta \pm 90^\circ$ , and all angles are measured from the  $x$  axis.

The photon source is chosen so that it emits two photons at essentially the same time with exactly opposite polarizations and in opposite directions. The original source of this type was a two-photon cascade from a calcium atom excited to its  $6S$  state, but the two subharmonic photons obtained in the nonlinear process called downconversion ( $\omega_{\text{in}} \rightarrow \omega_{\text{out}} + \omega'_{\text{out}}$ ) can also be arranged to have the desired properties.

In stage 1 of the experiment, the  $x$  channel of the first analyzer loop on the left is blocked and the  $\bar{\phi}$  channel of the last loop on the right is blocked [see Fig. 2(a)]. This arrangement means that the experimenter knows that all photons reaching the detector on the left must be  $y$ -polarized and, given the property of the source, the corresponding right-going photons are then guaranteed to be  $x$ -polarized. The experimenter records the fraction of times a photon is detected on the right, given the detection of a  $y$ -polarized photon on the left. This fraction will be designated as  $f(x, \phi)$  to indicate that the right-moving photon was originally  $x$ -polarized but was detected as  $\phi$ -polarized (necessarily so, because the  $\bar{\phi}$  channel was blocked).

In stage 2, the  $y$  channel of the first analyzer loop on the left is blocked and the  $\bar{\theta}$  channel of the middle loop on the right is blocked [see Fig. 2(b)]. This blocking means that the experimenter knows that all photons detected on the left must be  $x$ -polarized and, given the property of the source, the corresponding right-going photons are guaranteed to be  $y$ -polarized. The experimenter records the fraction of times a photon is detected on the right, given the detection of the  $x$ -polarized photon on the left. This fraction will be designated as  $f(y, \theta)$  to indicate that the right-moving photon was

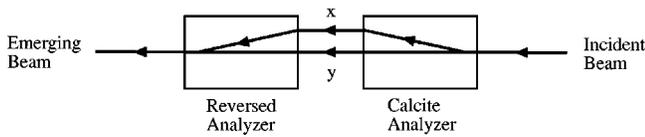


Fig. 1. Schematic picture of the two orthogonally polarized light channels in an analyzer loop, according to the conventions of French and Taylor (Ref. 8).

originally  $y$ -polarized but was detected as  $\theta$ -polarized (necessarily so, because the  $\bar{\theta}$  channel was blocked).

In stage 3, the  $\theta$  channel of the second analyzer loop on the left is blocked and the  $\bar{\phi}$  channel of the last loop on the right is blocked [see Fig. 2 (c)]. This arrangement means that the experimenter knows that all photons detected on the left must be  $\bar{\theta}$ -polarized and, given the property of the source, the corresponding right-going photons are guaranteed to be  $\theta$ -polarized. The experimenter records the fraction of times a photon is detected on the right, given the detection of the  $\bar{\theta}$ -polarized photon on the left. This fraction will be designated as  $f(\theta, \phi)$  to indicate that the right-moving photon was originally  $\theta$ -polarized but was detected as  $\phi$ -polarized (necessarily so, because the  $\bar{\phi}$  channel was blocked).

The total number of counts is assumed to be sufficient to make the three fractions that were detected in the three stages of the experiment statistically significant. The fractions are straightforwardly interpreted as photon-counting probabilities. The detection processes are assumed 100% efficient, which is not currently an absolutely airtight assumption.

### III. BELL INEQUALITIES

Three categories of photon counts are analyzed. In stage 1, the fraction of  $x$ -polarized photons transmitted through the three crystals on the right that are detected as  $\phi$ -polarized is  $f(x, \phi)$ . Because we do not ask which of the  $\theta$  or  $\bar{\theta}$  channels any of those photons went through in traversing the intermediate loop, we decompose  $f(x, \phi)$  to include both possibilities, which we indicate by writing  $f(x, \phi) = f(x, \theta, \phi) + f(x, \bar{\theta}, \phi)$ . In stage 2 we deal with the  $f(y, \theta)$  category,

which consists of all the right-going, originally  $y$ -polarized photons that emerge from the  $\theta$  channel, without caring if they went through the  $\phi$  or  $\bar{\phi}$  channels of the final analyzer loop, both of which are open. Thus a decomposition includes both of these possibilities:  $f(y, \theta) = f(y, \theta, \phi) + f(y, \theta, \bar{\phi})$ . The third category is made available by stage 3, in which we detect the fraction  $f(\theta, \phi)$  of right-going photons that began as  $\theta$ -polarized and emerged from the final crystal as  $\phi$ -polarized, without regard to whether they traversed the  $x$  or  $y$  channels of the first crystal. Obviously, in this case we can write  $f(\theta, \phi) = f(x, \theta, \phi) + f(y, \theta, \phi)$ .

By direct addition of the numbers of photons in the categories described, we see that

$$f(x, \phi) + f(y, \theta) = f(x, \theta, \phi) + f(x, \bar{\theta}, \phi) + f(y, \theta, \phi) + f(y, \theta, \bar{\phi}). \quad (1)$$

It is simple to observe that among the terms on the right-hand side of Eq. (1), we find both  $f(x, \theta, \phi)$  and  $f(y, \theta, \phi)$  and the sum of them is  $f(\theta, \phi)$ . That is, another way to write Eq. (1) is

$$f(x, \phi) + f(y, \theta) = f(\theta, \phi) + f(x, \bar{\theta}, \phi) + f(y, \theta, \bar{\phi}). \quad (2)$$

If we drop the two final terms (both are positive or zero fractions), we obtain the following inequality:

$$f(x, \phi) + f(y, \theta) \geq f(\theta, \phi), \quad (3)$$

which is an example of what is called a Bell inequality, after the physicist John Bell, who first studied their consequences in quantum physics beginning in the mid-1960s.

It is not difficult to see that a variety of inequalities that are similar to Eq. (3), but not exactly the same, can be derived equally easily. Another example is provided by  $f(x, \bar{\theta})$  and  $f(\theta, \bar{\phi})$ , because

$$f(x, \bar{\theta}) = f(x, \bar{\theta}, \phi) + f(x, \bar{\theta}, \bar{\phi}), \quad (4a)$$

$$f(\theta, \bar{\phi}) = f(x, \theta, \bar{\phi}) + f(y, \theta, \bar{\phi}). \quad (4b)$$

The sum gives  $f(x, \bar{\theta}) + f(\theta, \bar{\phi}) = f(x, \bar{\theta}, \phi) + f(y, \bar{\theta}, \bar{\phi}) + f(x, \theta, \bar{\phi}) + f(y, \theta, \bar{\phi}) = f(x, \bar{\phi}) + f(x, \bar{\theta}, \phi) + f(y, \theta, \bar{\phi})$ . By

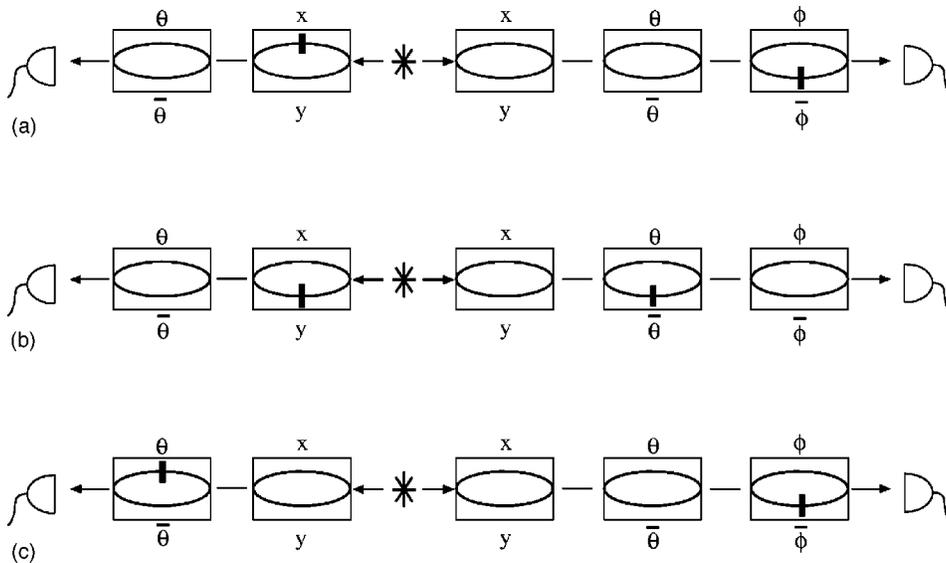


Fig. 2. Five analyzer loops make a Clauser-type experiment. (a)–(c) The channel-blocking called for in the three stages of the overall experiment, as described in the text.

dropping the last two terms, we obtain the inequality

$$f(x, \bar{\theta}) + f(\theta, \bar{\phi}) \geq f(x, \bar{\phi}). \quad (5)$$

Equation (5) is also a Bell inequality, as is  $f(y, \bar{\phi}) + f(x, \theta) \geq f(\theta, \bar{\phi})$ , which can be established in the same way.

It is important to repeat that such inequalities are not restricted to photons and their polarization or to similar properties of other quantum particles. As mentioned already, derivations of Bell inequalities have been provided for such disparate classes of objects as downhill skiers<sup>2</sup> and socks in a laundry.<sup>3</sup> None of these Bell inequalities are based on anything quantum mechanical. However, it is true that in the photon case, we did make a “quantum” concession by accepting photons as particles that can be counted. In this connection, it would be interesting for an undergraduate class to watch the movie, “Single Photon Polarization,”<sup>11</sup> where the existence of photons in a laser beam and their observable polarizations are demonstrated in the course of a laboratory experiment. In an unusual concession to an intended classroom audience, the movie action is mostly in real time and easy to follow.

#### IV. VIOLATION OF BELL INEQUALITIES

Now I will use the setup described above and the categories of data from the three stages of the Clauser-type experiment to show what is meant by a Bell violation. I will employ Malus’ law (from Malus’ observations of polarization<sup>12</sup> in Paris around 1808) to evaluate the various terms in our Bell inequalities. We will concentrate on the first Bell inequality in Eq. (3). Consider the first term. Only a fraction of the incoming  $x$ -polarized photons will find their way through a  $\phi$  polarizer, and according to Malus’ law, the fraction is  $\cos^2 \phi$ , if angles are measured with respect to the  $x$  axis, which is our convention. Malus’ law can also be applied to the other two terms appearing in Eq. (3). Hence, for the three terms, we find the following three results:

$$f(x, \phi) = \cos^2 \phi, \quad (6)$$

$$f(y, \theta) = \cos^2(90^\circ - \theta) = \sin^2 \theta, \quad (7)$$

$$f(\theta, \phi) = \cos^2(\phi - \theta), \quad (8)$$

and Bell’s inequality reads:

$$\cos^2 \phi + \sin^2 \theta \geq \cos^2(\phi - \theta). \quad (9)$$

Equation (9) applies to all values of  $\phi$  and  $\theta$ . Because no restriction has been placed on them, we are free to consider a simpler sub-case in which  $\phi = 2\theta$ . Then Eq. (9) becomes

$$\cos^2 2\theta \geq \cos^2 \theta - \sin^2 \theta = \cos 2\theta. \quad (10)$$

Because  $|\cos 2\theta| \leq 1$ , the trigonometric relation (10) is patently false for all values of  $\theta$  between  $0^\circ$  and  $45^\circ$ . In other words, the Bell inequality is violated for a wide choice of angles of observation of polarization, whereas the choice of angles made no difference to the derivation of the inequality.

Any violation of a Bell inequality in experimental data is important because the derivation of the Bell inequality itself had nothing to do with physics. It is a simple mathematical result obtained by an uncomplicated counting of members of objects in clearly defined categories. One might first think that a possible explanation for any violation lies with Malus’ law, which was the source of the trigonometric factors in Eq.

(9). But such a violation is very unlikely, because Malus’ law has never been found to fail in almost 200 years of careful observations of polarization.

#### V. IT MATTERS HOW WE COUNT

After deriving a Bell inequality and showing that it is not true, we need to understand how it happened. There is also a question in the background: where does quantum theory fit into our discussion? It hasn’t made an appearance yet, and so what does it mean to say that quantum theory violates Bell inequalities?

Careful reflection will show that the puzzling features of the Bell inequality (3) and its violation are associated with the first of the three terms in the inequality. We introduced it as

$$f(x, \phi) = f(x, \theta, \phi) + f(x, \bar{\theta}, \phi). \quad (11)$$

Equation (11) appears to be obviously true. We added the intermediate  $\theta$  and  $\bar{\theta}$  options because they exhaust all the possibilities—the analyzer does not allow the  $x$ -polarized photons to go anywhere else in the second stage than into either the  $\theta$  channel or the  $\bar{\theta}$  channel, both of which are open in the stage 1 experiment. Equation (11) thus asserts that  $f(x, \phi)$  gives the total count without regard to the intermediate channel chosen by any of the photons counted. However, the simplicity of our wording is deceptive. There are two different interpretations for the phrase “without regard to.” This phrase either means that (a) a distinction existed between the photons going through one or the other channel, and we ignored it in combining their counts, or that (b) no distinction existed that we could use to identify the photon channels separately, so there were no separate counts to be regarded.

In classical physics there is no meaningful difference between interpretations (a) and (b). Although classical analysts would have to agree that for interpretation (b), the fractions  $f(x, \theta, \phi)$  and  $f(x, \bar{\theta}, \phi)$  existed only in our mind and not in the experiment, they would still assert that a state of intermediate polarization was there to be measured. This assertion is the same as saying that because the photons must have gone through one or the other of the two channels, there must exist fractions representing the photon currents in the two channels. Another familiar expression of the same sentiment says that a falling tree makes noise, even if no one is available to hear it.

Quantum theory, on the other hand, is comfortable with the difference between the two meanings of “without regard to,” and even insists on the difference. Quantum theory is the modern theory of information. It insists that an apparatus, as it is set up and used, provides all the information that there is in an experiment. Quantum theory warns that adding extra information, however obvious, introduces a bias that can lead to error. That is what happened here. As the experiment was described, there were no data taken (just as important, there was no way to take data) that correspond to the fractions  $f(x, \theta, \phi)$  and  $f(x, \bar{\theta}, \phi)$ . Quantum theory says, therefore, there is no physical sense to an intermediate polarization. It is not an element of physical reality in this context, and not available for use in our analysis. That is, because our Bell inequality incorporates the experiment’s unobservable decomposition (11), the inequality has no standing at all in a quantum context.

## VI. SUMMARY

Quantum theory's insistence, just mentioned, should be emphasized when counting the contributions of quantum theory to our understanding of the natural world. One knows that quantum theory has a calculational framework attached to it that allows it to make specific predictions about physical events. These have been found to be universally correct to date, including predictions about Clauser-type polarization experiments. But apart from all its striking calculational successes, quantum theory brings more to physics, namely a world view that can be quite uncomfortable. To adopt an extreme phrasing, there really isn't a sound when a tree falls if there is no way to record it.

The student (and the typical physicist) reasonably thinks that there nevertheless is a sound, and this prejudice opens the door to a discussion of environmental interaction and decoherence, and of ways that an unobserved background can play the part of "apparatus" and permit realistic distinctions between initially entangled degrees of freedom. Decoherence is a subject of great current interest but too far afield to pursue here. Similarly, we have overlooked the way that one should react to the term "guaranteed" used in describing the three stages of the experiment, have side-stepped the opportunity to delve into the question of locality, and have not even mentioned attempts<sup>13</sup> to extend quantum theory so as to be compatible with realism, at the (fairly extreme) sacrifice of the positivity of probability.

## ACKNOWLEDGMENTS

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<sup>9</sup>An excellent instructional film showing the behavior of polarized light beams in an analyzer loop is "Interference in Photon Polarization," made for Education Development Center (Newton, MA).

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<sup>11</sup>An excellent instructional film using Malus' law to illustrate elements of single-photon counting physics is "Single Photon Polarization," by S. Berko, made for the Education Development Center (Newton, MA).

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