Continuous Frequency Entanglement: Effective Finite Hilbert Space and Entropy Control

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We examine the quantum structure of continuum entanglement and in the context of short-pulse down-conversion we answer the open question of how many of the uncountably many frequency modes contribute effectively to the entanglement. We derive a set of two-photon mode functions that provide an exact, discrete, and effectively finite basis for characterizing pairwise entanglement. Our analysis provides a basis for entropy control in two-photon pulses generated from down-conversion.

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Nonclassical properties of quantum states and their applications in communication have been a central topic in the field of quantum information [1]. It is known that quantum entanglement is the key to performing tasks that cannot be realized classically [2]. For this reason it is crucial to identify the methods of preparing and exploiting entangled systems, and particularly to develop quantitative methods for characterizing them. Among a few reliable sources of quantum entanglement, polarization entangled two-photon states generated from type-II phase-matched parametric down-conversion [3] have been used to demonstrate quantum teleportation [4,5], quantum dense coding [6], and entanglement swapping [7,8].

Although such two-photon down-conversion states have been widely used, their quantum features have not been fully exploited. Often these two-photon states are said to describe two particles, each with two internal states corresponding to the two polarizations. However, the influence of the vast Hilbert space associated with continuous field modes is just beginning to be investigated [9–13]. Indeed, a pair of entangled photons represents entanglement in a much broader class of systems, namely, continuously entangled quantum fields. Recently, Branning *et al.* have demonstrated a way to engineer quantum correlations by controlling spectral (frequency) amplitudes of two-photon pulses [14]. Their experiment highlights possibilities for applications in continuous-frequencies space. Note that these continuous degrees of freedom are accessible even in single photon or biphoton experiments.

Although the frequency continuum has infinitely many degrees of freedom, entanglement between two photons does not necessarily involve all of them. An open question is how to identify which degrees of freedom are active. The question is important because it is only when we understand how two photons are entangled that we may fully exploit them as quantum resources.

Here we answer this question by introducing a specific discretization of the continuum. This discretization is exact (not an approximation), unique, and useful. As an application to a realistic situation, we examine quantitatively the continuum entanglement of two-photon states generated in ultrashort pulse-pumped type-II down-conversion [15]. With specified experimental parameters, we determine the underlying discrete mode structures and the entropy of entanglement. Our approach is also applied to quantify outcomes of entropy control via phase-adjusted symmetrizations of two-photon amplitudes.

To begin, let us show, in a general form, a two-photon state vector:

$$|\Psi\rangle = \int \mathcal{A}(\omega_1, \omega_2)\hat{a}_H^\dagger(\omega_1)\hat{a}_V^\dagger(\omega_2)\,|0\rangle_H|0\rangle_V\,d\omega_1 d\omega_2\,. \tag{1}$$

Here $\hat{a}_H^\dagger(\omega_1)$ and $\hat{a}_V^\dagger(\omega_2)$ are creation operators with their subscripts indicating the (horizontal and vertical) polarizations. The function $\mathcal{A}(\omega_1, \omega_2)$ describes a two-photon amplitude for an H-polarized photon with frequency ω_1 and a V-polarized photon with frequency ω_2. In this paper we assume that both fields propagate in the same direction, so the direction labels are suppressed. This can be realized in down-conversion experiments in a collinear phase-matched configuration.

The fields associated with the two polarizations are entangled if $\mathcal{A}(\omega_1, \omega_2)$ cannot be factorized into a product of functions of ω_1 and ω_2. We may test the factorizability of $\mathcal{A}(\omega_1, \omega_2)$ by performing a Schmidt decomposition, i.e., by finding the eigenbasis of $\mathcal{A}$ such that it can be written as

$$\mathcal{A}(\omega_1, \omega_2) = \sum_n \sqrt{\lambda_n}\,\psi_n(\omega_1)\phi_n(\omega_2)\,, \tag{2}$$

where λ_n, ψ_n, and ϕ_n are solutions of the integral eigenvalue equations,

$$\int K_1(\omega, \omega')\psi_n(\omega')\,d\omega' = \lambda_n\psi_n(\omega)\,, \tag{3}$$

$$\int K_2(\omega, \omega')\phi_n(\omega')\,d\omega' = \lambda_n\phi_n(\omega)\,, \tag{4}$$

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with $K_1(\omega, \omega') \equiv \int \mathcal{A}(\omega, \omega_2)\mathcal{A}^*(\omega', \omega_2)\, d\omega_2$ and $K_2(\omega, \omega') \equiv \int \mathcal{A}(\omega_1, \omega)\mathcal{A}^*(\omega_1, \omega')\, d\omega_1$. Note that the kernels K_1 and K_2 can be interpreted as (one-photon) spectral correlations, and ψ_n and ϕ_n are their corresponding eigenfunctions. Obviously, $\mathcal{A}(\omega_1, \omega_2)$ is factorizable if there is only one term in the right side of Eq. (2). Therefore the two-photon state is entangled if the number of nonzero eigenvalues is more than one. We note that the use of the Schmidt decomposition for continuous variables has been discussed by several authors [10,16]. An important fact of the decomposition is that the eigenvalues λ_n are discrete, except for those at $\lambda = 0$. For a summary of mathematical details, see Ref. [16].

With the help of the decomposition (2), we define effective creation operators by

$$\hat{b}_n^\dagger \equiv \int \psi_n(\omega_1)\hat{a}_H^\dagger(\omega_1)\, d\omega_1\,, \tag{5}$$

$$\hat{c}_n^\dagger \equiv \int \phi_n(\omega_2)a_V^\dagger(\omega_2)\, d\omega_2\,. \tag{6}$$

From the orthogonality property of ψ_n and ϕ_n, it is easy to check that $\hat{b}_n$ and $\hat{c}_n$ satisfy the usual commutation relations for bosons: $[\hat{b}_n, \hat{b}_m^\dagger] = [\hat{c}_n, \hat{c}_m^\dagger] = \delta_{nm}$. In terms of $\hat{b}$'s and $\hat{c}$'s, the two-photon state takes the form

$$|\Psi\rangle = \sum_n \sqrt{\lambda_n}\, \hat{b}_n^\dagger \hat{c}_n^\dagger |0\rangle_H |0\rangle_V \tag{7}$$

in which the continuous integral (1) has been replaced by an infinite discrete summation. We emphasize that the discreteness is a consequence of the Schmidt basis and not an approximate box quantization.

The several advantages of using these unique "Schmidt-mode" operators now become clear. According to the description in Eq. (7), the two photons of different polarizations appear *only in pairs in the Schmidt modes*. If a photon is found in a b_n mode, then it is certain that the second photon must be in the c_n mode. In other words the decomposition identifies precisely in what modes photons go as pairs. The value of λ_n determines the occupation probability of the corresponding mode pair.

As we shall see below, although there are infinitely many modes in the fields, it may happen that only a few λ_n's are significant. Therefore the representation (7) also tells us *the effective dimension of the Hilbert space in which the state lives*. For example, if a state has only λ_1 and λ_2 nonzero, then the state can be written as $|\Psi\rangle = [\sqrt{\lambda_1}\, |1_H, 0_H\rangle |1_V, 0_V\rangle + \sqrt{\lambda_2}\, |0_H, 1_H\rangle |0_V, 1_V\rangle] \otimes |\text{vac}'\rangle$, where $|\text{vac}'\rangle$ denotes the vacuum state of modes other than the two. Therefore the representation (7) distinguishes which modes are in the vacuum state and then factors them out. As a result, we can describe the state in a lower-dimensional space. We remark that our present work is for systems with pure states. It remains an important issue as to how a Schmidt-type decomposition may be useful in analyzing the structure of mixed states, even for discrete systems [17].

The meaning of the b_n and c_n modes can be made more obvious if we define a field operator in the space-time domain ($c = 1$): $\hat{E}_\beta^{(+)}(x, t) \equiv \int \hat{a}_\beta(\omega)e^{-i\omega(t-x)}\, d\omega$ with $\beta = H, V$. We define "pulse modes" by Fourier transforms: $u_n(t) \equiv \int d\omega\, \psi_n(\omega)e^{-i\omega t}$, $v_n(t) \equiv \int d\omega\, \phi_n(\omega)e^{-i\omega t}$. By using the completeness of ψ_n and ϕ_n, the field operators take a simple form

$$\hat{E}_H^{(+)}(x, t) = \sum_n \hat{b}_n u_n(t - x)\,, \tag{8}$$

$$\hat{E}_V^{(+)}(x, t) = \sum_n \hat{c}_n v_n(t - x)\,. \tag{9}$$

Hence, when viewed in the time domain a photon in a b_n or a c_n mode refers to a single photon "traveling pulse" described by $u_n(t - x)$ or $v_n(t - x)$. We note that the expansion of the field operator in terms of pulse modes shares a similar idea with a temporal-mode analysis of light pulses generated in stimulated Raman scattering [18] or in two-photon spontaneous emission [10].

Now we show that the general method described can be applied to a case of practical interest: a two-photon state generated by down-conversion of an ultrashort pump pulse with type-II phase matching. By taking the down-conversion phase-matching condition into account, the two-photon state amplitude function is modeled by [19]

$$\mathcal{A}(\omega_o, \omega_e) = \alpha(\omega_o, \omega_e)\Phi(\omega_o, \omega_e)\,, \tag{10}$$

where $\alpha(\omega_o, \omega_e)$ is the spectral envelope and $\Phi(\omega_o, \omega_e)$ is the phase-matching function, neither of them factorable: $\alpha(\omega_o, \omega_e) = \exp[-(\omega_o + \omega_e - 2\bar{\omega})^2/\sigma^2]$, $\Phi(\omega_o, \omega_e) = \text{sinc}\{L[(\omega_o - \bar{\omega})(k_o' - k_p') + (\omega_e - \bar{\omega})(k_e' - k_p')]/2\}$, where σ is the pump spectral bandwidth. Note that we have used the subscripts o and e to relabel the polarizations (ordinary and extraordinary fields). Here k_o' and k_e' are the inverse of group velocities at the frequency $\bar{\omega}$, and k_p' is the inverse group velocity at the pump frequency $2\bar{\omega}$. In typical experiments the parameters have these values: $\sigma = 35\ \text{ps}^{-1}$, $(k_p' - k_e')L = 0.213$ ps, and $(k_p' - k_o')L = 0.061$ ps, corresponding to the crystal length $L = 0.8$ mm.

We have solved the integral eigenvalue equations (3) and (4) numerically by discretizing the kernels K_j. Convergent results are obtained if the infinite matrix is truncated at a sufficiently large size (800×800 in our case). With the parameters mentioned above, frequencies ranging $\pm 30\sigma$ around the mean frequency $\bar{\omega}$ are needed. In Fig. 1 we show the first ten values of λ_n. A contour plot of the two-photon amplitude function $\mathcal{A}$ is given in one of the insets. Our calculations show that the first six eigenvalues account for 96% of the state. *This indicates that the state vector is effectively in a remarkably low-dimensional space*. We have also calculated the quantity

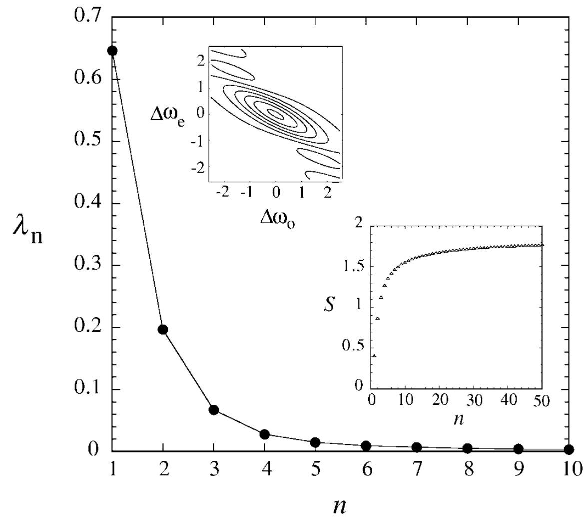

FIG. 1. The first ten eigenvalues λ_n. The left inset shows the contour plot of $\mathcal{A}$ in which $\Delta\omega_j = (\omega_j - \bar{\omega})$, $(j = o, e)$, in units of σ. The right inset shows S as a function of n. See text for the parameter values used. The connecting line is to guide the eye and has no other significance.

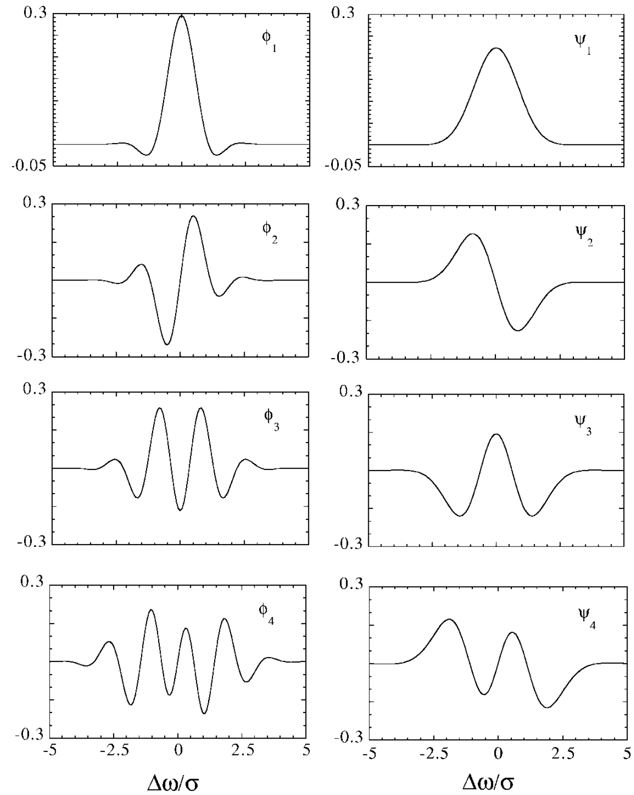

FIG. 2. The shape of the first two pairs of modes in frequency space for the state in Fig. 1.

$$S = \sum_{k=1}^{n} \lambda_k \log_2 \lambda_k \, , \tag{11}$$

which converges to the entropy of entanglement in the $n \to \infty$ limit. An inset of Fig. 1 shows how S converges as n increases. The first six modes contribute $S \approx 1.4$ which is already almost 80% of the large n limit $S \approx 1.8$.

In Fig. 2 we show the shape of the mode functions ψ_n and ϕ_n for $n = 1, 2, 3$, and 4. Because of the asymmetry of the ordinary and extraordinary fields, ψ_n and ϕ_n are not the same. The mode functions resemble the wave functions of a harmonic oscillator, which have the property that they are (essentially) their own Fourier transforms. Thus, our pulse modes in the time domain also have shapes similar to the spectral shapes in Fig. 2.

We point out that by decreasing the spectral bandwidth σ of the pump pulse, the entanglement can be increased. For example, in the case $\sigma = 17.5$ ps^{-1}, half of the bandwidth used above, we find that the entropy of entanglement can be increased to $S \approx 2.1$, compared with the value $S \approx 1.8$ shown in Fig. 1. Such an enhancement can be understood because a narrower bandwidth would imply a stronger correlation between the photons, as suggested in the function $\alpha(\omega_o, \omega_e)$.

However, we have discovered that more powerful control of S is obtained in an easily achieved experimental configuration mentioned by Branning *et al.* [14] in which the knowledge of the generation time of a photon pair is not degraded, as it is when the pump bandwidth is narrowed. We refer to phase-adjusted symmetry [14], with amplitudes $\mathcal{A}_\theta(\omega_o, \omega_e) \equiv \mathcal{A}(\omega_o, \omega_e) + e^{i\theta}\mathcal{A}(\omega_e, \omega_o)$. For simplicity we consider here only the symmetric and antisymmetric cases $\mathcal{A}_\pm$ corresponding to $\theta = 0$ and π. In the symmetric case we find that the entropy of entanglement is decreased to $S \approx 1.4$, smaller than the unsymmetrized case. This is due to interference effects which enhance occupation in the lowest mode, increasing the lowest mode probability to $\lambda_1 = 0.74$.

Interference is more drastic for the antisymmetric amplitude $\mathcal{A}_-$. In this case the eigenvalues λ_n form *degenerate pairs* (see Fig. 3). The value of the first eigenvalue $\lambda_1 = 0.27$ is much smaller than the original case. Because a larger number of modes are involved, the entropy of entanglement is larger. We see that $S \approx 2.9$, a value twice that in the symmetrized case. The inset of Fig. 3 shows a contour plot of $\mathcal{A}_-$. It is quite remarkable that interference causes more complicated features than those in Fig. 1. The lack of a single central peak indicates that the state is more nonfactorizable.

Note that, if the original polarizations are associated with distinct frequencies, i.e., if the original amplitude function $\mathcal{A}(\omega_1, \omega_2)$ is localized far away from the diagonal axis $\omega_1 = \omega_2$, one has $S \approx 0$. In this case (not realized in the experiment of [14]) the mere exercise of phase-adjusted symmetrization discussed above leads to a state with entanglement of $S \approx 1$. For example, a symmetrization of a bi-Gaussian $\mathcal{A}(\omega_1, \omega_2)$ localized far away from the diagonal would produce two symmetric bi-Gaussians with respect to the diagonal axis, independent of θ. In this case, since the two Gaussians are far apart, their

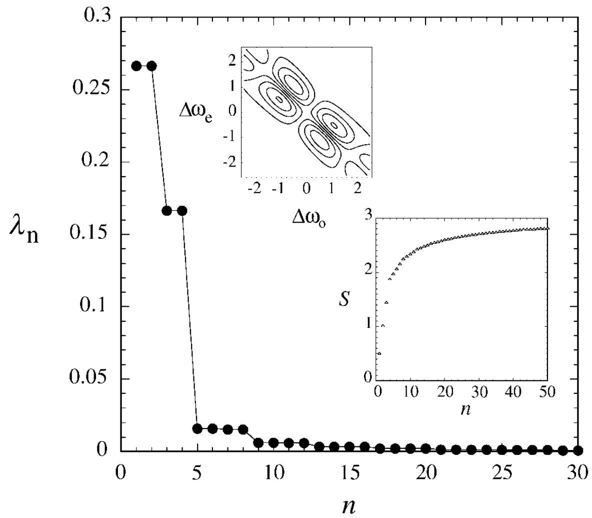

FIG. 3. Eigenvalues λ_n for the antisymmetrized two-photon amplitude. The left inset shows the contour plot of $\mathcal{A}_-$. The right inset shows S as a function of n. Same parameter values used as those in Fig. 1, and again the connecting line is only to guide the eye.

interference due to overlap can be neglected. Therefore symmetrization and antisymmetrization yield the same entanglement. However, if $\mathcal{A}(\omega_1, \omega_2)$ is localized on the diagonal axis, entanglement will strongly depend on interference as described above, and values well in excess of $S = 1$ can be achieved.

In summary, we first note that there are considerable advantages to exploiting continuous variables in quantum systems [20,21], and a recent demonstration includes the realization of quantum teleportation based on continuous field quadratures in squeezed light [22]. Here we have presented what we believe to be the first complete realization of entanglement involving the field-frequency continuum of the quantum electromagnetic field in an essential way. Infinitely many degrees of freedom are, in principle, accessible for information processing even when there are only two photons in the field. This is in contrast to the situation discussed by Braunstein and Kimble [20] and also fundamentally different from that of Zeilinger and co-workers' experiments [4,6].

We have shown that the Schmidt-mode basis gives a unique intrinsically pairwise entanglement, leading to an efficient representation for the system entropy. Our explicit calculations were made in the familiar context of type-II down-conversion, using experimentally realistic parameters. Our results show quantitatively for the first time the effectiveness of entropy control by experimental management of the phase symmetry of spectral correlations. Experimental determination of two-photon amplitudes is an important next step, and understanding how to measure Schmidt modes is the first task in this regard. We speculate that diffractive analogs of Stern-Gerlach spin filters will render the Schmidt modes directly accessible in the laboratory [23].

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