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Qubit cross talk and entanglement decay

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We study the decohering effect of qubit-qubit interactions on entanglement. Modeling such qubit cross talk by nearest-neighbor exchange interactions, we explore the Heisenberg spin chain as a system in which decoherence occurs dynamically, and we obtain an analytic description of the process. We find expressions for the short-time and long-time transfer and decay of entanglement in a long chain, and examine the role of entanglement propagation as a decoherence mechanism.

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I. INTRODUCTION

Qubit entanglement is an important resource for quantum computing.[1] One fundamental challenge is to generate specific forms of entanglement, such as entanglement between a selected pair of particles; another challenge is to avoid the influence of decoherence. Here we are interested in both of these challenges in the context of qubit interactions within a qubit array. Some qubit-qubit interactions may be completely managed by external manipulations and thus play a positive role in information processing and transfer, but it has to be expected that spontaneous, unmanaged interactions will also occur among the qubits. Such uncontrolled interactions will affect entanglement directly, leading to generalized diffusion of entanglement, usually best characterized simply as decay, but sometimes also involving spontaneous creation of entanglement. These processes can corrupt the calculations of a quantum computer.

Similarly, proposals for quantum teleportation[2] and quantum cryptography[3] often require that an entangled qubit pair be shared between ''Alice'', the sender, and ''Bob'', the receiver. If these qubits experience interactions with their environment, then the entanglement between the qubit pair will degrade, reducing the fidelity of the teleportation or communication. In such a situation the entangled pair and the environment may be considered as a qubit array.

To study the influence of qubit-qubit interactions, which we will refer to as entanglement cross talk, we employ a model in which cross talk occurs dynamically rather than stochastically. The understanding of stochastic influences such as thermal excitations is certainly important, but the action of qubits on each other via deterministic exchange interactions may be at least as significant as thermal noise, and more so in qubit assemblies where low temperatures are maintained. A number of specific qubit configurations have been proposed for various purposes, but none has been reduced to full-scale operating practice as yet, so rather than examine any specific prototype, we prefer a model that may be useful in understanding the influence of cross talk on entanglement in general.

Such a model is the Heisenberg spin chain (HSC), which has been studied for many years and whose dynamical properties are well known.[4] It can be considered as a system of interacting qubits, and it serves as a simple paradigm for studying the evolution of entanglement in a system subject to deterministic disruption of established entanglement by nearest-neighbor cross talk. We show that, at least in the HSC model, such decay of entanglement is subexponentially slow. The qubit crosstalk also causes the entanglement to flow along the spin chain, while deterministic decoherence causes the slow decay of this polarized entanglement current. This raises the possibility that Alice and Bob might exchange entanglement by means of the spin chain itself. The qubits in the HSC may be identified with a two-level electronic transition in an atom or with an exciton, or a nuclear or molecular spin, or any other two-state physical system that can couple to neighboring systems.

II. HSC MODEL

The Heisenberg spin chain consists of a linear lattice with N sites. Each site is occupied by a two-state quantum system, and we follow convention and adopt the nomenclature and notation of quantum angular momentum, treating each two-state system as though it were a spin. Given two orthogonal basis states $|a\rangle$ and $|b\rangle$ at each site, a conventional representation of the spin operator components takes the form

$$S^x=\frac{1}{2}(|a\rangle\langle a|+|b\rangle\langle a|),$$

$$S^y=\frac{i}{2}(|a\rangle\langle b|-|b\rangle\langle a|),$$

$$S^z=\frac{1}{2}(|b\rangle\langle b|-|a\rangle\langle a|). \tag{1}$$

It is easy to check that the Cartesian components of $\mathbf{S}$ defined in this way satisfy the conventional commutator relations for dimensionless angular momentum: $[S_x,S_y]=iS_z$, etc.

The Hamiltonian may then be written as

$$H=-\hbar g\sum_{i=1}^{N}\mathbf{S}_i\cdot\mathbf{S}_{i+1}. \tag{2}$$

Here $g>0$ is a coupling constant with dimensions of inverse time, and $\mathbf{S}_i$ is the spin operator for the ith spin. In the case of a closed chain we have $\mathbf{S}_{N+1}\equiv\mathbf{S}_1$ (periodic boundary conditions), while in the case of an open chain the upper limit of the summation must be replaced by $N-1$ (free boundary conditions).

The Hamiltonian (2) commutes with the total z-spin operator $S^z\equiv\Sigma_{i=1}^N S_i^z$, and therefore the eigenstates of the Hamiltonian have a specific number of spins flipped with respect to a reference state in which all spins are aligned. We choose as such a reference state $|\phi\rangle$, defined as the state with all spins down, i.e.,

$$|\phi\rangle\equiv|aa\cdots aa\rangle,\quad S^z|\phi\rangle=-\frac{N}{2}|\phi\rangle, \tag{3}$$

where $|a\rangle$ denotes a down spin and $|b\rangle$ will denote an up spin. This state is an element of the degenerate subspace of ground states of the Heisenberg model. We will call the subspace in which $S^z=-N/2+1$ the *single-excitation* states, although the states have varying energies and at least one such state is degenerate with the ground state (the model is gapless). Single-excitation states have particularly simple wave functions. If we let $|m\rangle$ denote a state in which the spin at site m is up and all others are down, i.e.,

$$|m\rangle\equiv|aa\cdots ab_m a\cdots aa\rangle, \tag{4}$$

then the energy eigenfunctions spanning the single-excitation subspace are

$$|\psi_\lambda\rangle=\frac{1}{\sqrt{N}}\sum_{m=1}^{N} e^{2\pi i\lambda m/N}|m\rangle, \tag{5}$$

for a closed chain, where λ is a quantum number ranging from 0 to $N-1$. The energies E_λ of these states are

$$E_\lambda+\frac{N}{4}\hbar g=\hbar g\left(1-\cos\frac{2\pi\lambda}{N}\right)\equiv\hbar\omega_\lambda, \tag{6}$$

where the constant term $-N\hbar g/4$ is the energy of the reference state. In the evolution of entanglement on the chain, inclusion of the ground-state energy simply contributes a fixed phase shift, which we will usually omit. The $\hbar\omega_\lambda$'s are the energies with respect to the reference state; we will refer to them as the chain energies. These results are usually derived via the Bethe ansatz.[5]

III. QUBIT CORRELATION

The primitive quantity relevant to entanglement cross talk in the qubit array is the single-excitation migration function. This is the quantum amplitude for finding an inverted spin displaced by r units to lattice site $i+r$ from initial site i after a (dimensionless) time $\tau\equiv gt$. If we define spin raising and lowering operators $S_i^\pm\equiv S_i^x\pm iS_i^y$ acting at site i, we can write the migration amplitude as a correlation, which we denote

$$\mathcal{C}(r;\tau)\equiv\langle\phi|S_{i+r}^-(\tau)S_i^+(0)|\phi\rangle, \tag{7}$$

for any i and (integral) displacement r. Here $\mathcal{C}(0,\tau)$ is simply the autocorrelation amplitude.

Since the Hamiltonian commutes with the total spin operator, the unitary evolution of this state will not take it out of the single-excitation subspace, and therefore we can propagate it using the "restricted" propagator

$$U(t_f;t_i)=\sum_{\lambda=0}^{N-1} e^{-i\omega_\lambda(t_f-t_i)}|\psi_\lambda\rangle\langle\psi_\lambda|. \tag{8}$$

Thus the migration amplitude is

$$\begin{aligned}\mathcal{C}(r;\tau)&=\langle\phi|S_{i+r}^-U(t;0)S_i^+|\phi\rangle\\&=\langle i+r|U(t;0)|i\rangle=\frac{1}{N}\sum_{\lambda=0}^{N-1} e^{i(2\pi r\lambda/N)-i\omega_\lambda t}.\end{aligned} \tag{9}$$

As expected, the result does not depend on the site i at which the spin was originally inverted or on the site $i+r$ which we examine at time τ, but only on the distance r between them. A global phase $e^{iN\tau/4}$ due to the ground-state energy $-\frac{1}{4}N\hbar g$ has been omitted.

The correlation can be evaluated approximately in the long-chain limit by converting it to an integral. After making the change of variables $z=2\pi\lambda/N$, approximating $(N-1)/N\approx1$, and recalling the definition (6) of ω_λ, we have

$$\mathcal{C}(r;\tau)\to\frac{e^{-i\tau}}{2\pi}\int_0^{2\pi} e^{irz+i\tau\cos z}dz=e^{-i\tau+i\pi r/2}J_r(\tau), \tag{10}$$

where $J_r(\tau)$ is the ordinary cylindrical Bessel function. Figure 1(a) shows the behavior of $|\mathcal{C}(r;\tau)|^2$ as a function of time τ, while Fig. 1(b) shows the behavior of $|\mathcal{C}(r;\tau)|^2$ as a function of lattice displacement r.

We can see that the correlation provides a direct estimate of the speed with which rearrangement propagates under the HSC exchange interaction. The well-known Bessel function behavior which is manifest in Fig. 1(a) shows that the expression $|\mathcal{C}(r;\tau)|^2=J_r^2(\tau)$ for the probability that the excitation has traveled a distance of r sites is very small until $\tau\approx r$ or $t\approx r/g$, after which it grows rapidly to its maximum. Thus, the intrinsic propagation speed is seen to be g sites per unit time, as might have been expected. We also note that in addition to the position-independent global phase $e^{-i\tau}$, the single-excitation migration function (10) has a position-dependent phase $e^{i\pi r/2}$. Therefore we can interpret the outcome of exciting a Heisenberg spin chain by inverting a single spin as a wave of excitation propagating in both directions along the chain from the initial site with velocity g and with a phase which rotates by $\pi/2$ per site as it progresses. However, the Bessel function is not non-negative and, therefore, cannot be interpreted strictly as a modulus. When $J_r(\tau)$ passes through a zero, the phase at r inverts. These effects will be important in interpreting the results of superposed excitations, since they lead to interference between excitation waves.

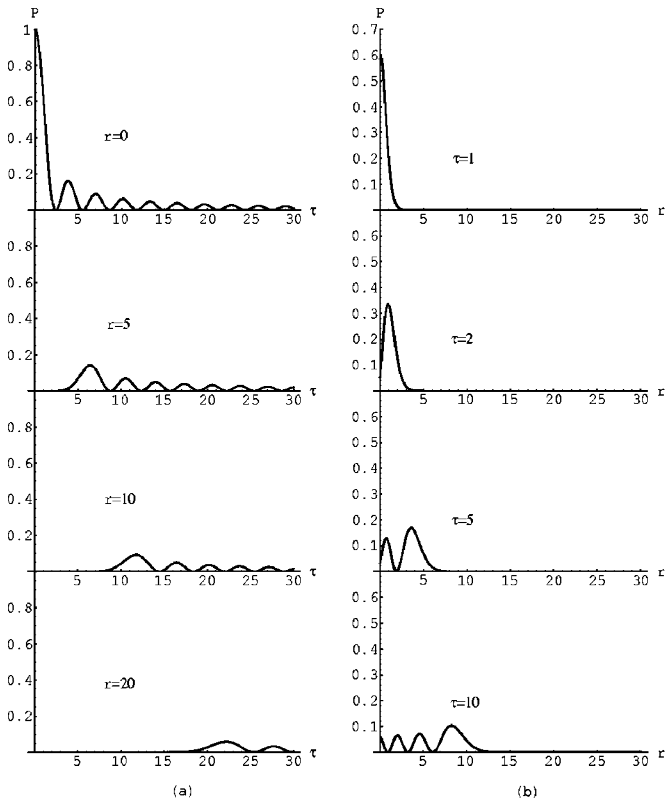

FIG. 1. Long-chain behavior of $|\mathcal{C}(r;\tau)|^2$ as given in Eq. (10), plotted (a) as a function of τ at various positions r and (b) as a function of r at various times τ.

IV. SPONTANEOUS GENERATION AND DECAY OF ENTANGLEMENT

We can use the above results, Eq. (9) or (10), to see that an initially unentangled state of the HSC evolves into an entangled one. Since the system's one-excitation eigenstates (5) are themselves entangled, this result is automatically a consequence of the time evolution, and in this sense one says that entanglement can arise spontaneously in the HSC, i.e., without the action of external forces.

First we illustrate the spontaneous generation of pairwise entanglement from a primitive single-excitation state. We begin with a single flipped spin at site 1 and evaluate the amplitude for later finding pairwise entanglement of arbitrary sites. Our notation for pairwise-entangled states will be

$$|\psi_{uv}^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}(|u\rangle \pm |v\rangle). \tag{11}$$

In a minimal array (only two qubits) these are obviously Bell states.

Thus we can evaluate $\langle \psi_{uv}^{+}|U(t;0)|1\rangle$ and find, as an example,

$$\langle \psi_{uv}^{+}|U(t;0)|1\rangle = \frac{1}{\sqrt{2}N}\sum_{\lambda=0}^{N-1} e^{-i\omega_\lambda t}(e^{i2\pi\lambda(u-1)/N}+e^{i2\pi\lambda(v-1)/N}), \tag{12}$$

which we approximate as

$$\langle \psi_{uv}^{+}|U(t;0)|1\rangle \to \frac{e^{-i\tau}}{\sqrt{2}}\times[e^{i\pi(u-1)/2}J_{u-1}(\tau)+e^{i\pi(v-1)/2}J_{v-1}(\tau)] \tag{13}$$

in the many-qubit limit, where the expression has been simplified by use of the identity $J_{-n}=(-1)^n J_n$. Clearly this is not identically zero, and so pairwise entanglement has been spontaneously generated. Of course, more complicated inseparable states are generated also. One moral of this is that entanglement is not a conserved quantity.

Once created, entanglement between a specific pair is dynamically unstable against cross talk, and the cross talk implied by the HSC Hamiltonian causes evolution of the qubit array that is equivalent to dispersion or diffusion or decay of entanglement. Consider, for example, the evolution of the nearest-neighbor entanglement $|\psi_{12}^{+}\rangle$. The probability $\mathcal{P}^{+}(1,2;1,2)$ for finding the HSC qubits in the same entangled state at a later time is

$$\mathcal{P}^{+}(1,2;1,2) \equiv |\langle \psi_{12}^{+}|U(t;0)|\psi_{12}^{+}\rangle|^2 = \left|\frac{2}{N}\sum_{\lambda=0}^{N-1}\cos^2\left(\frac{\pi\lambda}{N}\right)e^{-i\omega_\lambda t}\right|^2 \tag{14}$$

$$\to J_0^2(\tau)+J_1^2(\tau). \tag{15}$$

In Fig. 2 we compare the exact result (14) with the long-chain limit (15). As expected, the long-chain approximation is extremely accurate until about $\tau \approx N$, when the periodicity of the spin chain leads to the breakdown of the approximation. From the asymptotics of the Bessel functions[6] we see that in the long-chain limit entanglement decays nonexponentially as $\sim(1/\tau)$.

In the long-chain limit the entanglement decay (15) can easily be shown to be monotonic. This, however, is a special feature of the decay of the entanglement of an adjacent pair. More generally, we can consider decay of entanglement between nonadjacent sites. The corresponding probability $\mathcal{P}^{+}(u,u+r;u,u+r)$ for finding sites u and $u+r$ entangled at time τ given that they were perfectly $(+)$ entangled initially, i.e., that $|\psi(0)\rangle = |\psi_{uu+r}^{+}\rangle$, is

$$\mathcal{P}^{+}(u,u+r;u,u+r) = \left|\frac{2}{N}\sum_{\lambda=0}^{N-1}\cos^2\left(\frac{\pi\lambda r}{N}\right)e^{-i\omega_\lambda t}\right|^2 \to |J_0(\tau)+e^{i\pi r/2}J_r(\tau)|^2. \tag{16}$$

For $(-)$ entanglement, i.e., when $|\psi(0)\rangle = |\psi_{uu+r}^{-}\rangle$, we find that

$$\mathcal{P}^{-}(u,u+r;u,u+r) = \left|\frac{2}{N}\sum_{\lambda=0}^{N-1}\sin^2\left(\frac{\pi\lambda r}{N}\right)e^{-i\omega_\lambda t}\right|^2 \to |J_0(\tau)-e^{i\pi r/2}J_r(\tau)|^2. \tag{17}$$

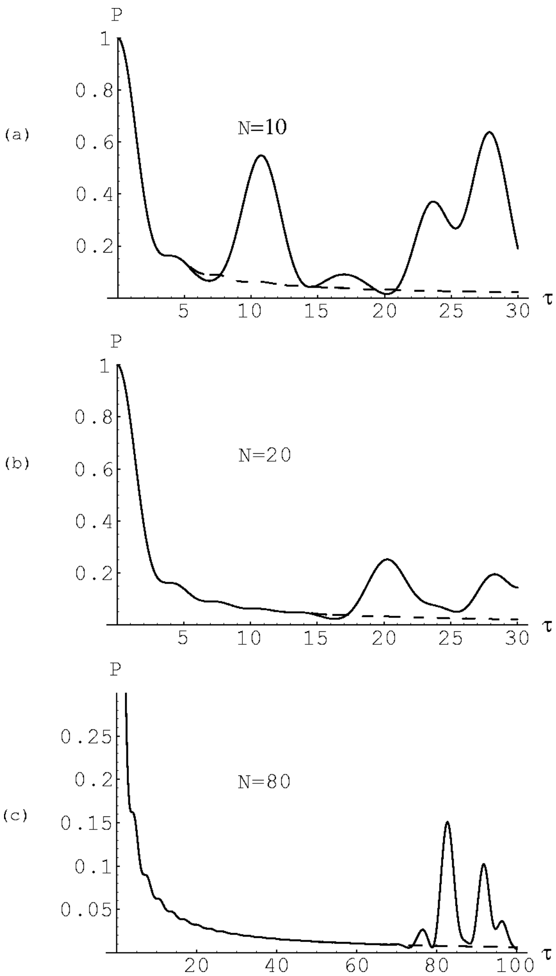

FIG. 2. Comparison of the exact [solid line, from Eq. (14)] and the long-chain [dashed line, from Eq. (15)] results for entanglement decay for (a) $N=10$, (b) $N=20$, and (c) $N=30$.

The effect of separation between sites (as measured by r) on entanglement decay is shown in Fig. 3. We see that even in the long-chain limit, the decay of entanglement between a nonadjacent pair is punctuated by partial revivals. We also

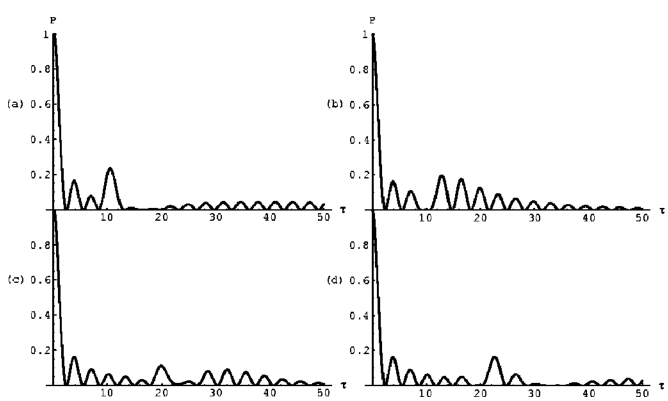

FIG. 3. Long-chain behavior of $\mathcal{P}^{+}(1,1+r;1,1+r)$ and $\mathcal{P}^{-}(1,1+r;1,1+r)$ as given in Eqs. (16) and (17), plotted as a function of τ: (a) $\mathcal{P}^{+}(1,11;1,11)$, (b) $\mathcal{P}^{-}(1,11;1,11)$, (c) $\mathcal{P}^{+}(1,21;1,21)$, and (d) $\mathcal{P}^{-}(1,21;1,21)$.

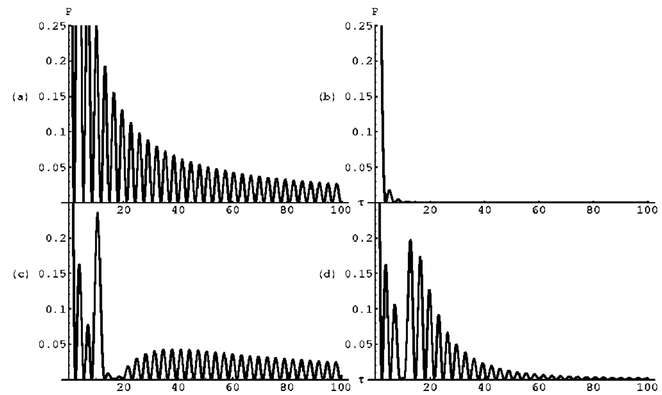

FIG. 4. Long-chain behavior of (a) $\mathcal{P}^{+}(1,3;1,3)$, (b) $\mathcal{P}^{-}(1,3;1,3)$, (c) $\mathcal{P}^{+}(1,11;1,11)$, and (d) $\mathcal{P}^{-}(1,11;1,11)$, as given in Eqs. (16) and (17), plotted as a function of τ.

note that for $(+)$ entanglement, there is a particularly prominent revival at $\tau \approx r$, while there is a corresponding minimum for $(-)$ entanglement.

The result above might be surprising, as it shows that $(+)$-entangled states and $(-)$-entangled states can behave quite differently in the HSC. From Eqs. (16) and (17) we see that

$$r \text{ odd:} \quad \mathcal{P}^{\pm}(u,u+r;u,u+r) \rightarrow J_0^2(\tau) + J_r^2(\tau), \tag{18}$$

$$r \text{ even:} \quad \mathcal{P}^{\pm}(u,u+r;u,u+r) \rightarrow J_0^2(\tau) + J_r^2(\tau) \pm 2(-1)^{r/2} J_0(\tau) J_r(\tau). \tag{19}$$

Recall that in Eq. (19), the $\pm$ corresponds to the sign in $|\psi^{\pm}_{uu+r}\rangle$. As another example of the importance of entanglement parity, we observe that when r is even, $\mathcal{P}^{+}$ exhibits the expected $\sim(1/\tau)$ asymptotic decay, but destructive interference causes $\mathcal{P}^{-}$ to decay much more rapidly, as shown in Fig. 4. This effect arises from the phase carried by the propagating excitation that results from inverting a spin. An initial state $|\psi^{+}_{uv}\rangle$ results in two propagating waves of excitation, one originating at u and one at v, differing in phase by $e^{i\pi(v-u)/2}$, while an initial state $|\psi^{-}_{uv}\rangle$ results in two similar waves, but differing in phase by $e^{i\pi(v-u)/2+i\pi}$. In this case, when $r \equiv v-u$ is odd, the phases are always in quadrature, and hence $\mathcal{P}^{\pm}$ is just the square of the hypotenuse of a right triangle whose sides have lengths $|J_0|$ and $|J_r|$. The phases remain in quadrature even when one or both Bessel functions (J_0 or J_r) pass through a zero and change sign. One the other hand, when r is even, the phases are parallel or antiparallel, depending on the choice of the initial phase (the $\pm$ in $|\psi^{\pm}_{uv}\rangle$), the displacement r, and the signs of the Bessel functions, which of course change with time. However, the result of these three factors is rather simple, as can be seen with a little asymptotic analysis. Using the standard relation[6] $J_n(\tau) \sim \sqrt{2/(\pi\tau)}\cos(\tau - \pi n/2 - \pi/4)$, we obtain, when r is even,

$$\mathcal{P}^{\pm} \sim \frac{2(2\pm 2)}{\pi\tau}\cos^2\left(\tau - \frac{\pi}{4}\right). \tag{20}$$

This clearly illustrates the destructive interference that occurs for $\mathcal{P}^-$. Thus qubit cross talk is sensitive to phase. A similar analysis explains the occurrence of maxima for $\mathcal{P}^+$ and minima for $\mathcal{P}^-$ at $r \approx \tau$ in Fig. 3.

Thus the HSC exhibits a deterministic form of decoherence. Schemes for the teleportation of quantum states[2] and for quantum cryptography[3] commonly utilize an entangled qubit pair shared between two parties, referred to as Alice and Bob. Equations (16) and (17) represent a limitation on the ability of Alice and Bob to store pairwise qubit entanglement in the presence of exchange interactions. The decay of pairwise entanglement is, however, only one manifestation of this decoherence. As we will see, the evolution of a pairwise entangled state creates creates a propagating wave of entanglement on the Heisenberg spin chain, but the same dynamical evolution that propagates pairwise entanglement also attenuates the entanglement, and so decoherence in effect competes with propagation.

V. GUIDING ENTANGLEMENT

Suppose we invert our viewpoint and think of cross talk as a control mechanism to be used to generate entanglement between a specific pair (say, u and v) at a given time. Then an initial state such as $|m\rangle$ is not optimal. We should instead select $U(0;t)|\psi_{uv}^{\pm}\rangle$, which is just the desired state propagated back to the initial time. But in general this state is highly nonlocal, involving all the spins on the chain. Suppose that we can manipulate only a few spins of the chain. To what extent can we create the desired entangled state, starting from a *local* entangled state?

For example, perhaps the the qubits at sites 1 and 2 are addressable (say by a laser) so that we may set their initial state, while the remainder of the spin chain remains unexcited. If we maximally entangle them, so that $|\psi(0)\rangle = |\psi_{12}^{+}\rangle$, we find that at a later time the probability $\mathcal{P}^+(u,v;1,2)$ for finding pairwise $(+)$ entanglement between *any* sites u and v is

$$\begin{aligned}\mathcal{P}^+(u,v;1,2) &\equiv |\langle \psi_{uv}^{+}|U(t;0)|\psi_{12}^{+}\rangle|^2\\ &= \left|\frac{1}{2N}\sum_{\lambda=0}^{N-1}\left(e^{i2\pi(u-1)\lambda/N}+e^{i2\pi(u-2)\lambda/N}\right.\right.\\ &\quad \left.\left.+e^{i2\pi(v-1)\lambda/N}+e^{i2\pi(v-2)\lambda/N}\right)e^{-i\omega_\lambda t}\right|^2\\ &\to \frac{1}{4}\left|i^{(u-1)}J_{u-1}(\tau)+i^{(u-2)}J_{u-2}(\tau)\right.\\ &\quad \left.+i^{(v-1)}J_{v-1}(\tau)+i^{(v-2)}J_{v-2}(\tau)\right|^2. \qquad (21)\end{aligned}$$

One way to monitor the evolution of entanglement is to study how adjacent pairs elsewhere on the spin chain become entangled as a function of their displacement r from the original pair. The probability for this is $\mathcal{P}^+(u,u+1;1,2)$, which is illustrated in Fig. 5:

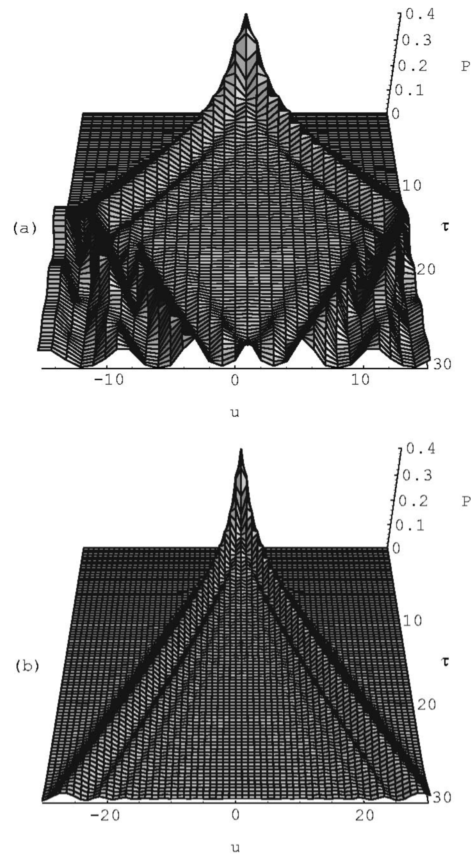

FIG. 5. (a) Exact behavior of $\mathcal{P}^+(u,u+1;1,2)$ as given in Eq. (22) for $N=31$ plotted as a function of u and τ. (b) Long-chain behavior of $\mathcal{P}^+(u,u+1;1,2)$ as given in Eq. (23) plotted as a function of u and τ.

$$\begin{aligned}\mathcal{P}^+(u,u+1;1,2) &\equiv |\langle \psi_{uu+1}^{+}|U(t;0)|\psi_{12}^{+}\rangle|^2\\ &= \left|\frac{1}{2N}\sum_{\lambda=0}^{N-1}\left(2e^{i2\pi(u-1)\lambda/N}+e^{i2\pi(u-2)\lambda/N}\right.\right.\\ &\quad \left.\left.+e^{i2\pi u\lambda/N}\right)e^{-i\omega_\lambda t}\right|^2 \qquad (22)\\ &\to \frac{1}{4}|J_u(\tau)-2iJ_{u-1}(\tau)-J_{u-2}(\tau)|^2. \qquad (23)\end{aligned}$$

We see that, in effect, cross talk causes pairwise entanglement to propagate along the spin chain. The decoherence that qubit cross talk also induces manifests itself in the gradual loss of amplitude of the entanglement signal, but the pulse itself propagates essentially without broadening. This represents a mechanism by which entanglement generated between two qubits can be transferred to a remote pair of qubits, where it might be recovered by, for example, a projection measurement on the spin chain. We might think of the source qubits, 1 and 2 in this example, as being controlled by one person—say, the famous cryptographer Alice—and the target qubits u and $u+1$ being owned by her cohort Bob. Then $\mathcal{P}^+(u,u+1;1,2)$ is simply the probability that Alice transfers the entanglement to Bob successfully.

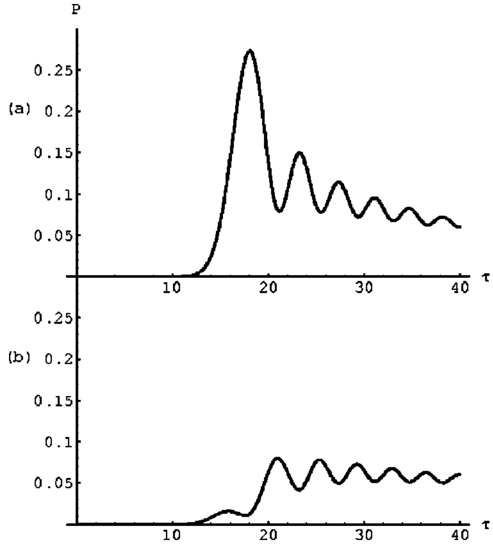

FIG. 6. (a) "Burst" of entanglement around $\tau=18$ between sites 17 and 18 on an $N=32$ spin chain, from Eq. (22). (b) This burst is not present between other pairs of sites, such as 16 and 17 (illustrated here) on the same ($N=32$) chain.

This probability can be optimized by a felicitous choice of position for Bob's qubit pair. While in the long-chain limit Bob simply wants his qubits to be as near Alice's as possible, for a finite chain there is a particularly prominent revival when the counterpropagating pulses reach the opposite side of the closed chain. If 1 and 2 are the initially entangled pair, then sites $N/2+1$ and $N/2+2$ experience a sudden burst of entanglement probability around $\tau=N/2$, as illustrated in Fig. 6(a). For comparison, the adjacent pair $N/2$ and $N/2+1$ [shown in Fig. 6(b)] experiences a much more modest increase in entanglement. This height of this burst is dependent on the length of the chain; it also depends on the parity of N and on the sign [$(+)$ or $(-)$] of the entanglement. These effects can be accounted for by considering the phase relations between the propagating waves, as was done for entanglement decay above.

Given an initial state such as $|\psi_{12}^{+}\rangle$, cross talk causes the qubit at site 1 to become entangled with remote qubits even as its entanglement with qubit 2 decays. The probability $\mathcal{P}^{+}(1,v;1,2)$ measures the extent to which entanglement between sites 1 and 2 has been shifted to entanglement between sites 1 and v:

$$\mathcal{P}^{+}(1,v;1,2)\equiv|\langle\psi_{1v}^{+}|U(t;0)|\psi_{12}^{+}\rangle|^2$$
$$=\left|\frac{1}{2N}\sum_{\lambda=0}^{N-1}(1+e^{-i2\pi\lambda/N}+e^{i2\pi(v-1)\lambda/N}+e^{i2\pi(v-2)\lambda/N})e^{-i\omega_\lambda t}\right|^2 \tag{24}$$
$$\to\frac{1}{4}|J_0(\tau)+iJ_1(\tau)+i^{(v-1)}J_{v-1}(\tau)+i^{(v-2)}J_{v-2}(\tau)|^2. \tag{25}$$

While Eqs. (22) describes the probability of transferring a pairwise entangled state to a remote pair of qubits, Eq. (24) describes the probability of distributing entanglement, so that Alice retains one member of the entangled pair while Bob receives the other. Such distributed entanglement is the key to quantum teleportation[2] and to some[3] forms of quantum cryptography. Initially $\mathcal{P}^{+}(1,v;1,2)$ is always 1/4. This transient entanglement dies away quickly, leaving a propagating pulse, which represents the probability that Alice's qubit 1 and Bob's qubit 2 are entangled.

VI. DISCUSSION

We have examined the consequences of nearest-neighbor cross talk in a linear qubit array, using the Heisenberg spin chain as the context for illustration and calculations, but focusing only on the lowest excitations possible within the chain. Several consequences have been noted, including the spontaneous generation of entanglement as well as the striking differences in evolution of initially symmetric and antisymmetric entanglement pairings. Specific examples were given of the evolution of pairwise entanglement within the single-excitation subspace. When the qubit array is sufficiently long for the ends to be remote, elementary formulas were obtained for propagation of entanglement, and a specific expression was obtained for the propagation velocity. Decohering effects due to nearest-neighbor exchange were found to be subexponential in their temporal behavior. The possibility of tailoring entanglement packets was pointed out. Examination of higher-order qubit entanglements will be reported elsewhere.

ACKNOWLEDGMENTS

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