

# Entanglement of internal and external angular momenta of a single atom

Ashok Muthukrishnan<sup>1</sup> and C R Stroud Jr

The Institute of Optics, University of Rochester, Rochester, NY 14627, USA

E-mail: amuthuk@physics.tamu.edu

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## Abstract

We consider the exchange of spin and orbital angular momenta between a circularly polarized Laguerre–Gaussian beam of light and a single atom trapped in a two-dimensional harmonic potential. The radiation field is treated classically but the atomic centre-of-mass motion is quantized. The internal levels in the atom are taken to be Rydberg circular states. The spin and orbital angular momenta of the field are individually conserved upon absorption, with the result that the electronic and motional degrees of freedom of the atom are entangled. We suggest applications in quantum information processing.

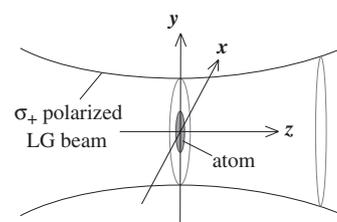
**Keywords:** Laguerre–Gaussian laser beam, orbital angular momentum, atomic centre-of-mass quantization, angular momentum entanglement, quantum information processing

## 1. Introduction

The Laguerre–Gaussian (LG) laser modes are known to possess well defined, discrete values of orbital angular momentum per unit energy [1]. The orbital angular momentum of the field is distinct from the spin angular momentum associated with the polarization state of the field. In the paraxial limit, the orbital angular momentum is polarization independent [2] and arises solely from the azimuthal phase dependence of the field mode which gives rise to helical wavefronts.

The interaction of LG modes with atoms has been studied extensively in the classical limit of the atom as a point particle [3]. It has been shown that the atom experiences a torque from the radiation pressure force, which transfers the angular momentum from the laser beam to the atom. This effect has been indirectly observed in the nonlinear four-wave mixing of LG modes in a cold atomic sample [4]. There have also been proposals to use LG modes to create vortices in Bose–Einstein condensates [5, 6], where the orbital angular momentum is transferred from the laser beam to the vortex trap state.

In this paper we consider the interaction of a circularly polarized LG mode with a *single* trapped atom whose centre-of-mass (CM) motion is quantized in the plane perpendicular to



**Figure 1.** The atom is prepared in the vibrational ground state by a collinear trapping beam (not shown). A left-circularly polarized LG mode couples the electronic and centre-of-mass angular momenta of the atom. The unit vectors  $x$  and  $y$  define the transverse plane of the beam waist.

the beam (see figure 1). In the paraxial limit, we know that the angular momentum of the atom-plus-field system is separately conserved for spin and orbital components [7]. That is, the selection rules for circular polarization determine the change in the atom's internal magnetic quantum number, while the orbital angular momentum of the field is directly transferred to the atomic centre-of-mass wavefunction. We use this mechanism to entangle the internal and external angular momenta of the atom.

Quantum entanglement in this context is of a different kind than that introduced in the landmark paper by Einstein, Podolsky and Rosen [8], where the observables that are en-

<sup>1</sup> Now at Department of Physics, Texas A & M University, College Station, TX 77843, USA.

tangled belong to two *separated* particles. In our case, the entangled observables belong to the *same* atom, and describe respectively its internal and external state. This is similar to the entanglement that was produced between the electronic and vibrational degrees of freedom of an ion in a linear ion trap [9]. In our case, the proposed entanglement occurs in the angular momentum observable rather than in energy.

To describe the external angular momentum of the atom as a good quantum number, we consider a cylindrically symmetric trap potential. The atom may be confined in the transverse plane by an optical dipole trap created by another LG beam that has a doughnut-shaped transverse profile and is blue-detuned to the atomic transition [10]. The confinement results from the radial null in the field intensity along the beam axis. The polarization and order of the LG mode can then be changed during the entanglement phase while maintaining the same beam axis. For simplicity, we take the trap potential to be radially harmonic in the transverse plane.

In section 2, we introduce the basis states for the centre-of-mass and electronic degrees of freedom of the atom. The electronic states are taken to be Rydberg circular states [11], which have the maximum angular-momentum component  $m\hbar$  along the  $z$ -axis for a given principal quantum number  $n = m + 1$ . These states are extremely long lived, with lifetimes that scale as  $n^5$  and are of the order of  $10^{-2}$  s for  $n = 30$ . We assume that the trap frequency  $\nu$  is large compared with the electronic linewidths, which is essential to have selective excitation of the trap states.

In section 3 we derive the Hamiltonian that describes the interaction of the atom with a circularly polarized LG beam. The quantization of the atomic position is made in the limit that the size of the centre-of-mass wavefunction is small compared with the beam waist. This linearizes the interaction Hamiltonian and simplifies the calculation. In section 4 we show how the interaction leads to an entanglement between the internal and external states of the atom. Finally in section 5, we consider the relevance of this phenomenon for quantum information applications.

## 2. Basis states

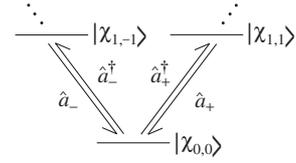
The Hamiltonian for a harmonic trapping potential in two dimensions is

$$\hat{H}_{\text{CM}} = \frac{1}{2m}(\hat{P}_X^2 + \hat{P}_Y^2) + \frac{1}{2}m\nu^2(\hat{X}^2 + \hat{Y}^2) \quad (1)$$

where  $m$  is the mass of the atom and  $\nu$  is the radial trap frequency. We use capital letters  $X$  and  $Y$  to denote the centre-of-mass coordinates. For simplicity, assume that the atom is tightly confined along the trap axis,  $\nu_z \gg \nu$ , using additional laser beams. The Hamiltonian in equation (1) describes two independent one-dimensional harmonic oscillators along the Cartesian axes defining the trap plane, with corresponding annihilation operators

$$\hat{a}_X = \frac{1}{\sqrt{2}}\left(\frac{\hat{X}}{R_0} + i\frac{\hat{P}_X}{\hbar/R_0}\right) \quad (2)$$

$$\hat{a}_Y = \frac{1}{\sqrt{2}}\left(\frac{\hat{Y}}{R_0} + i\frac{\hat{P}_Y}{\hbar/R_0}\right) \quad (3)$$



**Figure 2.** Ground and first excited states of the trap. The four transitions are shown, with corresponding operators.

where  $R_0 = \sqrt{\hbar/m\nu}$  sets the scale for the radial size of the trap. Since we are interested in the angular momentum of the atom, it is convenient to introduce the operators

$$\hat{a}_{\pm} = \frac{1}{\sqrt{2}}(\hat{a}_X \mp i\hat{a}_Y) \quad (4)$$

which serve to raise ( $\hat{a}_{+}^{\dagger}$  and  $\hat{a}_{-}$ ) and lower ( $\hat{a}_{-}^{\dagger}$  and  $\hat{a}_{+}$ ) the angular momentum  $L_z$  along the trap axis. We can show that

$$\hat{L}_Z = \hbar(\hat{n}_{+} - \hat{n}_{-}) \quad (5)$$

where  $\hat{n}_{+} = \hat{a}_{+}^{\dagger}\hat{a}_{+}$  and  $\hat{n}_{-} = \hat{a}_{-}^{\dagger}\hat{a}_{-}$  count the number of right and left circular quanta respectively. The Hamiltonian for the two-dimensional oscillator becomes

$$\hat{H}_{\text{CM}} = \hbar\nu(\hat{n}_{+} + \hat{n}_{-} + 1). \quad (6)$$

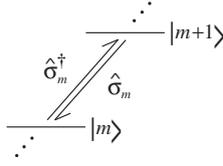
The centre-of-mass eigenstates of the atom can be written in terms of the energy and angular-momentum quantum numbers,  $N = n_{+} + n_{-}$  and  $M = n_{+} - n_{-}$ . For a fixed value of  $N \geq 0$ , there are  $N + 1$  degenerate angular-momentum states for which  $M = -N, -N + 2, \dots, N$ . The ground state and the first excited states of the trap in polar coordinates,  $R = \sqrt{X^2 + Y^2}$  and  $\Phi = \tan^{-1}(Y/X)$ , are given by the wavefunctions

$$\chi_{0,0}(R, \Phi) = \frac{1}{R_0\sqrt{\pi}} \exp\left(-\frac{R^2}{2R_0^2}\right) \quad (7)$$

$$\chi_{1,\pm 1}(R, \Phi) = \frac{1}{R_0\sqrt{\pi}} \left(\frac{R}{R_0}\right) \exp\left(-\frac{R^2}{2R_0^2} \pm i\Phi\right). \quad (8)$$

The energy levels for these states and the corresponding transition operators are shown in figure 2. The amplitudes of  $\chi_{1,\pm 1}$  are shaped in the form of a doughnut with a null at the centre, and the azimuthal phase is determined by the angular momentum  $M = \pm 1$ . In general, the  $\chi_{N,M}$  wavefunctions are also given by Laguerre–Gaussian modes [12]. That is, the LG modes describe *both* the atomic centre-of-mass probability amplitude and the laser field amplitude on the trap plane, which are physically distinct quantities.

To describe the internal angular momentum of the atom, we introduce a basis of circular electronic states. Since each circular state has the maximum value of the magnetic quantum number,  $m = l = n - 1$ , only neighbouring states are coupled according to dipole selection rules. This makes them good candidates for atomic two-level systems (or qubits). More generally, the circular-state basis forms a ladder of states, one in each Rydberg manifold  $n$ , each coupled to its neighbours by circularly polarized light. We start out by using the full basis to describe the atomic observables, and specialize to the case of a two-level system in the next section.



**Figure 3.** Atomic circular states  $|m\rangle = |n = m + 1, l = m, m\rangle$ . The two transition operators are shown.

We use small letters to denote internal degrees of freedom in the atom. The internal angular momentum in the circular-state basis  $|m\rangle = |n = m + 1, l = m, m\rangle$  is given by

$$\hat{l}_z = \sum_{m=0}^{\infty} m\hbar |m\rangle\langle m|. \quad (9)$$

A left-circularly polarized field causes  $\Delta m = \pm 1$  transitions between neighbouring states. The atomic dipole moment  $\hat{\mathbf{d}} = \hat{x}\mathbf{x} + \hat{y}\mathbf{y} + \hat{z}\mathbf{z}$  can be written as

$$\hat{\mathbf{d}} = \frac{1}{2} \sum_m d_m [(x + iy)\hat{\sigma}_m + (x - iy)\hat{\sigma}_m^\dagger] \quad (10)$$

where  $d_m = e(x - iy)_{m,m+1} = e(x + iy)_{m+1,m}$  is the dipole-moment matrix element, and  $\hat{\sigma}_m = |m\rangle\langle m+1|$  is the lowering operator for the transition between neighbouring circular states. As figure 3 shows, the effect of  $\hat{\sigma}_m^\dagger$  and  $\hat{\sigma}_m$  is to raise and lower the internal atomic angular momentum in a given state.

### 3. Interaction Hamiltonian

The laser field that couples the electronic and motional degrees of freedom of the atom is described by a scalar LG mode  $u_{lp}$ . These modes are characterized by an angular momentum  $l\hbar$  about the propagation axis, and have  $p + 1$  radial nodes in the transverse intensity distribution [1]. For simplicity, we consider the  $p = 0$  case, which corresponds to a doughnut-shaped intensity distribution for  $l \neq 0$ . The transverse profile of the mode at the beam waist  $w_0$  is given by

$$u_l(R, \Phi) = \mathcal{E}_l \left(\frac{R}{w_0}\right)^{|l|} \exp\left(-\frac{R^2}{w_0^2} + il\Phi\right) \quad (11)$$

where  $R$  and  $\Phi$  are the same polar coordinates that describe the centre-of-mass position of the atom in section 2.

We are interested in the limit in which the size of the centre-of-mass wavefunction  $R_0$  is small compared with the radius of the LG mode. This affords a linearization of the interaction Hamiltonian analogous to the Lamb–Dicke limit in trapping and cooling. In this limit, we are justified in expanding the LG mode in powers of  $R/w_0$ ,

$$u_l(R, \Phi) = \mathcal{E}_l \left(\frac{R}{w_0}\right)^{|l|} \exp(il\Phi) + \mathcal{O}[(R/w_0)^{|l|+2}]. \quad (12)$$

Keeping only the leading-order term in this expansion, we quantize the atomic centre-of-mass position as follows. For  $l = \pm|l|$ ,

$$\left(\frac{\hat{R}}{w_0}\right) \exp(\pm il\Phi) = \frac{\hat{X} \pm i\hat{Y}}{w_0} = \eta(\hat{a}_\pm^\dagger + \hat{a}_\mp) \quad (13)$$

where  $\eta = R_0/w_0$  compares the size of the trap ground state with the beam waist. When  $\eta = 0$ , the atom behaves like a point particle. When  $\eta \ll 1$ , we can treat the interaction Hamiltonian to lowest order in the centre-of-mass position operators for the atom.

To couple neighbouring circular states in the atom, we need a left-circularly polarized field. Using the truncated form of the LG mode  $u_l$  in equation (12), and using the quantization condition in equation (13) for  $l = \pm|l|$ , we write the transverse electric field as

$$\begin{aligned} \mathbf{E}_l &= u_l(\hat{R}, \hat{\Phi})(\mathbf{x} + i\mathbf{y}) \exp(-i\omega t) + \text{h.c.} \\ &\approx \mathcal{E}_l \eta^{|l|} (\hat{a}_\pm^\dagger + \hat{a}_\mp)^{|l|} (\mathbf{x} + i\mathbf{y}) \exp(-i\omega t) + \text{h.c.} \end{aligned} \quad (14)$$

where h.c. denotes the hermitian conjugate. It is important to note that the laser field in equation (14) is *not* a quantized field, in the sense that we did not introduce creation and annihilation operators for the field. However, the mode functions of the classical field depend on the centre-of-mass coordinates of the atom, which are quantized in the interaction. This explains why the classical field in equation (14) depends on quantum operators.

The coupling between the LG field mode and the trapped atom is described by the  $\mathbf{d} \cdot \mathbf{E}$  Hamiltonian. Using the dipole moment and the field vector from equations (10) and (14), we find that for  $l = \pm|l|$ , the interaction Hamiltonian is given by

$$\begin{aligned} \hat{H}_{\text{int}} &= -\hat{\mathbf{d}} \cdot \mathbf{E}_l \\ &= -\frac{\hbar}{2} \sum_m \eta^{|l|} \Omega_{m,l} (\hat{a}_\pm^\dagger + \hat{a}_\mp)^{|l|} \hat{\sigma}_m^\dagger \exp(-i\omega t) + \text{h.c.} \end{aligned} \quad (15)$$

where  $\Omega_{m,l} = 2d_m \mathcal{E}_l / \hbar$  is the Rabi frequency, and we have used the vector identities  $(\mathbf{x} \pm i\mathbf{y}) \cdot (\mathbf{x} \pm i\mathbf{y}) = 0$  and  $(\mathbf{x} \pm i\mathbf{y}) \cdot (\mathbf{x} \mp i\mathbf{y}) = 2$ .

We now specialize to the case of a two-level system formed by two neighbouring circular states  $m$  and  $m + 1$ . In the interaction picture, the states evolve only according to  $H_{\text{int}}$ . The atomic operators evolve as  $\hat{\sigma}_m(t) = \hat{\sigma}_m \exp[-i(\omega_{m+1} - \omega_m)t]$ , where  $\omega_m$  are the atomic frequencies. Similarly, the centre-of-mass operators evolve as  $\hat{a}_\pm(t) = \hat{a}_\pm \exp(-i\nu t)$ , where  $\nu$  is the trap frequency. Consider the situation where the field is tuned to the  $|l|$ th sideband below the atomic resonance,

$$\omega = (\omega_{m+1} - \omega_m) - |l|\nu. \quad (16)$$

In the rotating-wave approximation, we ignore counter-rotating terms in the interaction Hamiltonian and are left with only the two circular states  $m$  and  $m + 1$  contributing to the sum in equation (15). Furthermore, if the field is sufficiently narrow in spectrum compared with the trap frequency, only the  $|l|$ th power of the operators  $\hat{a}_\pm$  contributes to the interaction in equation (15), assuming  $\eta^{|l|} \Omega_{m,l} \ll \nu$ , and we can ignore the cross terms. The interaction Hamiltonian simplifies to

$$\hat{H}_{\text{int}} = -\frac{\hbar}{2} \eta^{|l|} \Omega_{m,l} \hat{a}_\mp^{|l|} \hat{\sigma}_m^\dagger + \text{h.c.} \quad (17)$$

To interpret the interaction in physical terms, recall that  $\hat{a}_+^\dagger$  and  $\hat{a}_-$  raise the centre-of-mass angular momentum, while  $\hat{a}_+^\dagger$  and  $\hat{a}_+$  lower it. This can be seen from equation (5). Similarly,  $\hat{\sigma}_m^\dagger$  ( $\hat{\sigma}_m$ ) raises (lowers) the internal angular momentum of the atom in the two circular states. Thus, the Hamiltonian in

equation (17) clearly shows that the orbital angular momentum of the LG mode is transferred to the external angular momentum of the atom, while the spin angular momentum associated with circular polarization is transferred to the internal angular momentum of the atom. The spin and orbital components are separately conserved in the paraxial limit.

Choosing the orbital angular momentum of the LG mode to be positive or negative,  $l = \pm|l|$ , correlates the change in the internal and external angular momenta of the atom,  $\Delta m$  and  $\Delta M$ . For example, when  $l < 0$  the external angular momentum is raised whenever the internal angular momentum is lowered, and vice versa. The field frequency in equation (16) corresponds to tuning to the  $|l|$ th sideband on the *lower* side of the atomic resonance. This choice governs the parity of the transitions between the centre-of-mass states as shown in figure 2, and correlates the change in the energy and angular momentum of the trap,  $\Delta N$  and  $\Delta M$ .

#### 4. Entanglement

We use equation (17) as the starting point for a discussion of quantum entanglement between the internal and external angular momenta of the atom in the trap. Consider  $l = -1$ , which gives the left-circularly polarized LG field a net angular momentum of zero. In this case, the changes in internal and external angular momenta of the atom are equal in magnitude but opposite in sign. The time evolution operator is given by

$$\begin{aligned} \hat{U}_{\text{int}}(t) &= \exp(-i\hat{H}_{\text{int}}t/\hbar) \\ &= \exp\left[i\frac{\eta t}{2}\left(\Omega_{m,-1}\hat{a}_+\hat{\sigma}_m^\dagger + \Omega_{m,-1}^*\hat{a}_+^\dagger\hat{\sigma}_m\right)\right]. \end{aligned} \quad (18)$$

Consider the action of this operator on the state  $|m\rangle|\chi_{0,0}\rangle$ . Since the internal angular momentum of the atom can only be raised by  $\hat{\sigma}_m^\dagger$ , the external angular momentum has to be lowered by  $\hat{a}_+$ . However  $|\chi_{0,0}\rangle$  is the lowest-energy state of the trap and cannot be further reduced in energy. Hence the state  $|m\rangle|\chi_{0,0}\rangle$  does not evolve in time according to this interaction. This restriction does not apply to the state  $|m+1\rangle|\chi_{0,0}\rangle$ , since the atom is in the higher-angular-momentum circular state to begin with, and we find Rabi oscillations between states  $|m+1\rangle|\chi_{0,0}\rangle$  and  $|m\rangle|\chi_{1,1}\rangle$ . To summarize, we find that

$$\hat{U}_{\text{int}}(t)|m\rangle|\chi_{0,0}\rangle = |m\rangle|\chi_{0,0}\rangle \quad (19)$$

$$\begin{aligned} \hat{U}_{\text{int}}(t)|m+1\rangle|\chi_{0,0}\rangle &= \cos(\eta\Omega t/2)|m+1\rangle|\chi_{0,0}\rangle \\ &+ ie^{-i\phi}\sin(\eta\Omega t/2)|m\rangle|\chi_{1,1}\rangle \end{aligned} \quad (20)$$

where we have defined  $\Omega_{m,-1} = \Omega e^{i\phi}$ .

Equations (19) and (20) give the basic ingredients for quantum control of the selected internal and external states of the atom. When the trap is in the ground state and the atom is prepared in a coherent superposition of the circular states  $m$  and  $m+1$ , a  $\pi$ -pulse transfers this coherence to the centre-of-mass state of the atom in the trap,

$$\begin{aligned} \hat{U}_{\text{int},\pi} : [c_m|m\rangle + c_{m+1}|m+1\rangle]|\chi_{0,0}\rangle \\ \mapsto |m\rangle[c_m|\chi_{0,0}\rangle + c_{m+1}|\chi_{1,1}\rangle] \end{aligned} \quad (21)$$

where we have chosen  $\phi = \pi/2$ . Alternately, if the atom is in the upper circular state  $m+1$  and the trap is in the ground

state, a  $\pi/2$ -pulse creates maximal entanglement between the internal and external states,

$$\hat{U}_{\text{int},\pi/2} : |m+1\rangle|\chi_{0,0}\rangle \mapsto \frac{1}{\sqrt{2}}[|m+1\rangle|\chi_{0,0}\rangle + |m\rangle|\chi_{1,1}\rangle] \quad (22)$$

where  $\phi = \pi/2$  again. We have to be in the adiabatic limit where the pulse length is long enough that the spectrum does not overlap neighbouring trap states in energy.

Equation (22) is the main result of this paper, that we can in principle generate states of a single atom that are entangled in internal and external angular momenta using a circularly polarized LG mode. This is a new form of entanglement that relies on the conservation of angular momentum rather than energy. The two observables that are entangled are  $L_z$  and  $l_z$ , defined in equations (5) and (9), respectively.

The experimental difficulty is in measuring the quantized centre-of-mass state of the atom in the trap. A direct observation of the trap state may be engineered as follows. When the atoms are released from an excited trap state  $|\chi_{1,1}\rangle$ , they escape with a net linear momentum in the azimuthal direction, which may be detected by time-of-flight measurements using a suitably positioned detector array. An indirect observation of the entanglement present in equation (22) is possible using a weak probe pulse resonant with the circular states  $m+1$  and  $m+2$ . In this case, only the state  $|m+1\rangle|\chi_{1,1}\rangle$  is affected by the pulse, and the absorption of a photon would distinguish this state from  $|m+1\rangle|\chi_{0,0}\rangle$ .

#### 5. Discussion

It is intriguing to consider the application of the ideas in this paper to quantum information processing. Rydberg circular states have long decoherence times due to spontaneous emission. The two circular states  $m$  and  $m+1$  may be thought of as a qubit, and the interaction with the LG mode provides a controlled coupling to the second qubit formed by the ground and first excited states of the trap. In this context, equations (19) and (20) allow arbitrary two-qubit unitary operations, where the internal state of the atom is transformed conditional on the trap state.

One possibility to scale up this scenario is to consider two or more atoms individually trapped and manipulated in this manner. A coupling between two atoms may be achieved by entangled photons in LG modes, as demonstrated recently in the parametric down-conversion experiment of [13]. The trap states of each atom can play the role of an auxiliary qubit that mediates the quantum logic between computational qubits formed by the internal states of the atoms. Decoherence issues involved with trapping and cooling the atom to the centre-of-mass ground state benefit from the weak coupling of neutral atoms to the environment.

Lastly, we highlight the benefits of going beyond two internal states (beyond qubits) in the atom. We chose the circular states because they made a good two-level system. However, there are  $n^2$  angular-momentum states in the atom for each principal quantum number  $n$ , all of which are degenerate in hydrogen. A single field that is resonant with two Rydberg manifolds simultaneously couples all pairs of angular-momentum states that satisfy the selection rules. This

allows for the possibility of *multilevel* entanglement of the internal atomic state with its centre-of-mass motion. Angular-momentum control in this regime is particularly suited for large-scale information processing in atoms.

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