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Spontaneous-noise entanglement and photon wave functions

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Abstract

We undertake a pure-state analysis of a noise-dominated quantum event, namely spontaneous photon emission by an excited atom. While pure, the final state is nonseparably entangled. We calculate the participation ratio that provides a measure of the nonseparability, in the context of Schmidt-type analysis. The Schmidt modes serve as pairwise "pointer" bases in the joint Hilbert space, and suggest a nonlocal joint control of the photon and atom as well as the existence of localized photon wave functions.

Nonseparability of states can lead to a type of nonlocal control of one quantum system by another. For example, it is well known by now that the nonlocality associated with nonseparable quantum states is the key to performing computational tasks that cannot be realized classically [1]. Entanglement of the two photons emitted in spontaneous parametric down conversion, for example, has been examined in both contexts: photon localization [2], and quantum information [3,4], and a combination of these contexts in quantum pattern formation [5].

In this paper we examine a different instance of entanglement arising from quantum noise, i.e., random single-photon emission of an excited atom in free space, where the absence of electromagnetic confinement (no mirrors, waveguides, cavities, bandgap structures, etc.) guarantees continuous degrees of freedom for the emitted photon. Then conservation of momentum enforces a quantum correlation that entangles the momenta of the photon and the recoiling atom. Recently the first experiment on photon–atom correlation in atomic spontaneous emission was reported [6], and further experimental work in this fundamental area can be expected.

The spontaneous radiative decay of interest is sketched in Fig. 1. Let $\vec{p}$ and $\vec{r}$ denote the center of mass momentum and position of the atom with mass m and resonance frequency ω_0, and let σ_j denote the atomic operators in the usual Pauli matrix notation. Then the Hamiltonian of the system under the rotating wave approximation is given by

$$H = \frac{(\hbar\vec{p})^2}{2m} + \int d^3k\,\hbar\omega_k a_k^\dagger a_k + \frac{1}{2}\hbar\omega_0(\sigma_z + 1) + \hbar\int d^3k\,(g(\vec{k})\sigma_- a_k^\dagger e^{-i\vec{k}\cdot\vec{r}} + \text{h.c.}). \tag{1}$$

Here a_k and $a_k^\dagger$ are the annihilation and creation operators of the plane-wave field mode with frequency $\omega_k = ck$, satisfying $[a_k, a_{k'}^\dagger] = \delta^3(\vec{k} - \vec{k}')$. The strength of the radiative coupling between the atom and the field is given by

$$\hbar g(\vec{k}) = \sqrt{\frac{\hbar\omega_k}{2\epsilon_0(2\pi)^3}}\, e\vec{r}_0 \cdot \vec{\epsilon_k}, \tag{2}$$

with $\vec{\epsilon_k}$ and $e\vec{r_0}$ denoting the polarization vector of the field and the transition dipole moment of the atom respectively.

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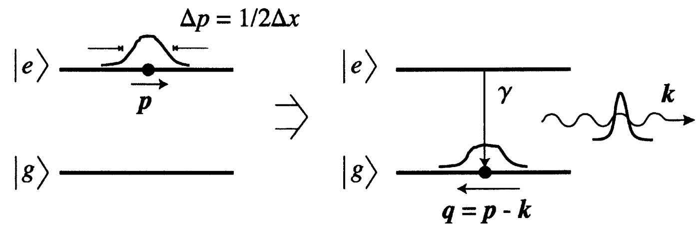

Fig. 1. Spontaneous emission of a photon from a two-level atom with an initial momentum distribution $\vec{p}$. The atom gains recoil momentum $\hbar\vec{q}$ by emitting a photon of wave vector $\vec{k}$.

The atom is prepared in the excited state at $t = 0$, and no photons are present. At a later time t we denote an emitted photon wave vector by $\vec{k}$, and the photon–atom state is of the form

$$|\Psi(t)\rangle = \int d^3p\, a(\vec{p}, t)|e; 0; \vec{p}\rangle + \int d^3k \int d^3p\, b(\vec{p}, \vec{k}, t)|\vec{p} - \vec{k}, g; 1_k\rangle. \tag{3}$$

The exact expressions for the amplitudes $a(\vec{p}, t)$ and $b(\vec{p}, \vec{k}, t)$ were found by Rzążewski and Żakowicz [7]. We have $a(\vec{p}, t) = a_0(\vec{p})e^{-i(E_p/\hbar+\omega_0-i\gamma)t}$ and

$$b(\vec{p}, \vec{k}, t) = \frac{g(\vec{k})a_0(\vec{p})}{(E_{p-k} - E_p)/\hbar + kc - \omega_0 + i\gamma}\{e^{-i(E_p/\hbar+\omega_0-i\gamma)t} - e^{-i(E_{p-k}/\hbar+kc)t}\}, \tag{4}$$

where $E_p \equiv (\hbar\vec{p})^2/2m$. Here the radiative halfwidth γ has been obtained by the usual Weisskopf–Wigner method.

After a long time ($\gamma t \gg 1$) the system state vector becomes

$$|\Psi(t)\rangle \to \int d^3k \int d^3q\, C(\vec{q}, \vec{k})e^{-i(E_q/\hbar+kc)t}|g; \vec{q}; 1_k\rangle, \tag{5}$$

where we have changed the variable such that the final momentum for the atom $\vec{q} \equiv \vec{p} - \vec{k}$ is shown explicitly. The amplitude $C(\vec{q}, \vec{k})$ is given by

$$C(\vec{q}, \vec{k}) = \frac{g(\vec{k})a_0(\vec{q} + \vec{k})}{(E_q - E_{q+k})/\hbar + kc - \omega_0 + i\gamma}. \tag{6}$$

The absolute square of $C(\vec{q}, \vec{k})$ tells us the probability of finding the atom having a momentum $\vec{q}$ and the emitted photon with the wave vector $\vec{k}$. We choose a frame in which the atom is initially at rest on average and we describe its positional uncertainty by a Gaussian distribution in momentum: $a_0(\vec{p}) = a_0(|\vec{p}|) = \exp[-(p/\Delta p)^2]$, so that the atom is prepared as a wave packet with momentum spread Δp.

Given finite Δp, the photon and atom do not have to recoil back to back, but we will restrict our attention to this case and use the subscript π to denote this restriction. Then the photon + atom state is given by the product of the electronic ground state and an entanglement of the center of mass with the emitted photon:

$$|\Psi_\pi\rangle = |g\rangle \otimes \int\int dq\, dk\, C_\pi(q, k)|q\rangle \otimes |1_k\rangle, \tag{7}$$

and since the entanglement is the interesting feature we will drop $|g\rangle$ from further consideration. We can approximate $C_\pi(q, k)$ as

$$C_\pi(\delta q, \delta k) = \frac{N \exp[-(mc^2\delta q/\hbar\omega_0\Delta p)^2]}{\delta q + \delta k + \frac{\hbar\omega_0^2}{2mc^2} + i\gamma}, \tag{8}$$

where we have used inequalities characteristic of optical transitions and atoms: $mc^2 \gg \hbar\omega_0 \gg \hbar\gamma$. The amplitude peaks at $kc \approx \omega_0$, so we have defined δq and δk by $qc = \omega_0 + (mc^2/\hbar\omega_0)\delta q$ and $kc = \omega_0 + \delta k$. Resonance (energy conservation) occurs at $\delta k \approx 0$. On the other hand, because of the factor $\hbar/2m$ in the denominator, q takes a broad range of values limited only by Δp. Even after these approximations the joint state of the atom and the photon are nonseparably entangled since $C_\pi(q, k)$ cannot be factored into a product of a function of $\vec{q}$ alone times a function of $\vec{k}$ alone.

Generally a quantum-mechanical system consisting of two subsystems A and B can be described by a two-body wave function of the form $|\psi_{AB}\rangle = \sum_{i,j} C_{ij}|a_i\rangle \otimes |b_j\rangle$, where $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$ are the bases for the two subsystems. However, if the composite system is in a pure state: $\hat{\rho} = |\psi_{AB}\rangle\langle\psi_{AB}|$, then there exist paired orthonormal bases in the two subspaces, $\{|u_i\rangle\}$ in A and $\{|v_j\rangle\}$ in B such that

$$|\psi_{AB}\rangle = \sum_i g_i |u_i\rangle \otimes |v_i\rangle. \tag{9}$$

This is known as the Schmidt decomposition of the two-particle state [3,4,8], and we will use it to write the photon–atom amplitude $C_\pi(q,k)$ in the form

$$C_\pi(q,k) = \sum_n \sqrt{\lambda_n}\psi_n(q)\phi_n(k), \tag{10}$$

where $\psi_n(q)$ and $\phi_n(k)$ are the (unique) Schmidt mode functions associated with the atom and the photon respectively.

The element of control that we mentioned above is now evident. A canonical (projective) measurement of one of the degrees of freedom, say of the atom since it is detectable with higher efficiency, gives immediate and very specific information about the photon, without detecting it and even if it is far away. The uniqueness of the Schmidt expansion is the key to this control.

There are other advantages to the expansion. For example, the photon–atom amplitude is now represented in terms of a discrete-mode basis, which introduces the useful countability property to the quantum description, not available with the original continuous mode description. Note that the countability comes without needing to "quantize" the description by hand, i.e., we have not introduced artificial periodic or box boundary conditions at any point, but have used only the fact that the Schmidt functions are eigenfunctions of density matrices, which, as operators with finite trace, have only discrete eigenvalues. Furthermore, the value of λ_n conveniently determines the occupation probability of the nth mode pair.

The participation number K has been introduced [9] to provide a normalized quantitative measure of the degree of entanglement:

$$K \equiv \frac{1}{\sum_n \lambda_n^2} \tag{11}$$

As soon as one can count the modes, one can estimate the strength of the entanglement by the number of Schmidt mode pairs that participate significantly in the decomposition. A disentangled (factorable) state has only one nonzero λ (i.e., one pair of Schmidt modes) and it gives $K = 1$, and if there are N pairs of modes participating equally, $K = N$. In the general case K defines the number of mode pairs significantly involved.

We have evaluated the Schmidt modes for a particular choice of atomic and photonic parameters, and some results are shown in Fig. 2 for the value $\eta = 10$, where $\eta \equiv (\hbar\omega_0\delta_p/mc\gamma)$ is the parameter that effectively controls the entanglement. Its origin in the theory is revealed if we rewrite the exponential in (8) in condensed form as $\exp[-(\delta q/\eta)^2]$. We have found that η is a kind of "observability index", and an intuitive interpretation of it can be given [10]. Similarly, an

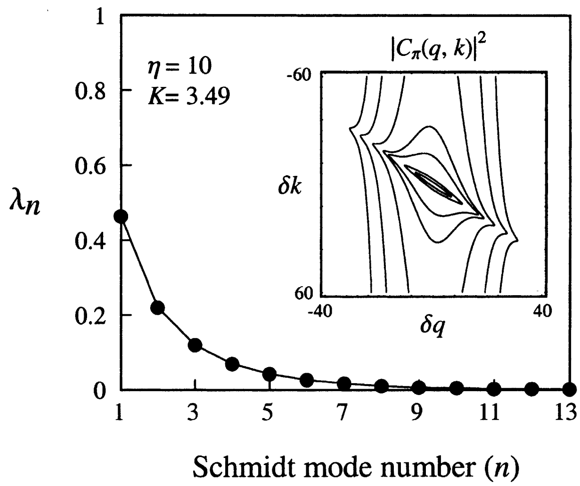

Fig. 2. Distriubtion of the eigenvalues λ_n for $\eta = 10$. The inset shows a contour plot of the absolute square of amplitude $C_\pi(q,k)$, where the dimensionless axis labels δq and δk are defined in the text.

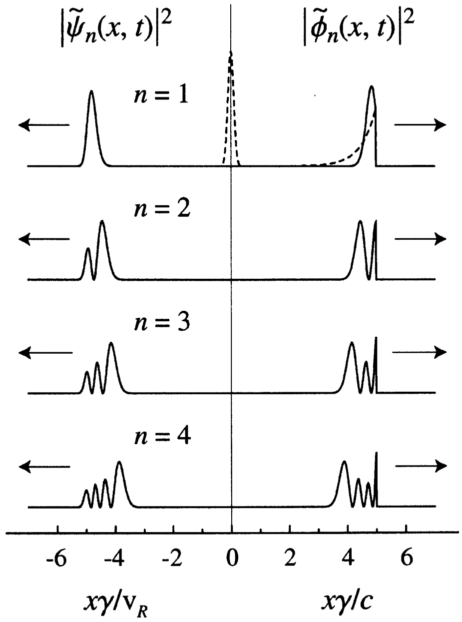

Fig. 3. The first four Schmidt pairs in position space at the time $t = 5\gamma^{-1}$. The left column gives the atom modes and the right column gives the photon modes, with arrows indicating the propagation directions. Here $\eta = 10$ and $\omega_0/\Delta p = 20$ are used. The spatial axes of the atom and photon modes are in different dimensionless units. The mean recoil velocity of the atom $\hbar\omega_0/mc = \eta\gamma/\Delta p$ is denoted by V_R. The dotted lines in the first row show the atom and photon states in the zero recoil limit. They are plotted using the same scale as the entangled case.

analysis of the Schmidt modes themselves shows that they can be interpreted as photonic wave functions, and examples are shown in Fig. 3.

In conclusion, we have used the Schmidt mode analysis to examine the pure state arising in the vacuum noise-initiated process of spontaneous photon emission. We have found it possible to quantify the entanglement, and in the case that $\eta = 10$ there are remarkably few modes engaged. There is less justification than one is usually taught, to say that the spontaneous emission process has access to a continuum of free-space modes. In effect, the traditional use of these modes does not reveal the restrictions that recoil entanglement places on the photon and atom. As a final remark, since the K parameter measures the number of effective states, it is conceptually connected to the notion of entropy S. In this sense we have shown that the recoil entanglement creates entropy from pure quantum dynamics.

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