The union of physics and information

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Abstract

A union of quantum mechanics, thermodynamics and information theory is presented. It is accomplished by reinterpreting the mathematical formalism of Everett's many-worlds theory of quantum mechanics and augmenting it to include preparation according to a given ensemble. The notion of *directed entanglement* is introduced through which both classical and quantum communication over quantum channels are reduced to entanglement transfer. The paradox of constant thermodynamic entropy in a closed quantum system is resolved.

The view taken in this Letter is that the totality of conceptual experience can be described in terms of correlated random variables. This will allow us to make contact with Shannon's information theory [1] in which random variables are the carriers of information. Two protagonists sharing the same physical world is no more than classical correlations between the states of their knowledge regarding that world. Similarly, the observation of definite physical laws is no more than classical correlations between states of knowledge regarding two consecutive acts of measurement, or preparation and measurement, depending on the experiment. For instance, a ball kicked by Alice seen as obeying Newton's deterministic laws of motion is merely a statement about the correlation between her knowledge of its initial velocity (preparation) and that of its position when it hits the ground (measurement).

We present a mathematical model in accord with these principles that strikes a balance between standard Copenhagen quantum mechanics, where the wavefunction is not intended to describe reality but merely to represent our knowledge of it, and Everett's many-worlds interpretation [2] which attempts to include the observer as just another physical system, leading to the unresolved basis problem [3]. In our model the universe $\mathcal{U}$ is divided into subsystems which can either belong to the set of "physical" entities $\mathcal{P}$ or to the set of instances of knowledge $\mathcal{K}$. Mathematically, it is intimately related to Everett's many-worlds theory, with the distinction that we do not imbue $\mathcal{K}$ with a "physical" interpretation; it merely keeps track of the conceptual experience of the observer/preparer in relation to what has been observed/prepared. The set $\mathcal{K}$ is divided into subsets associated with particular protagonists, such as $\mathcal{K}_{Alice}$ and $\mathcal{K}_{Bob}$. At any given time the universe is described by a pure state $|\Psi\rangle_{\mathcal{U}}$. The state of some subsystem A is described by the density matrix obtained from $|\Psi\rangle_{\mathcal{U}}$ by tracing out the rest of the universe $\mathcal{U}/A$:

$$\rho_A = tr_{\mathcal{U}/A}|\Psi\rangle\langle\Psi|_{\mathcal{U}}. \quad (1)$$

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The Hilbert space of $\mathcal{U}$ is constantly being augmented by new instances of knowledge, initially in some fixed pure state (although they immediately get entangled with already existing memebers of $\mathcal{U}$; as pure states they serve no function). Otherwise, $|\Psi\rangle_{\mathcal{U}}$ can only evolve from one moment to the next according to some unitary operator U

$$|\Psi\rangle_{\mathcal{U}} \longrightarrow U\,|\Psi\rangle_{\mathcal{U}}, \tag{2}$$

possibly entangling the different subsystems. These are the building blocks of the theory. Now we illustrate how measurement, preparation and communication are described in terms of it.

(i) **Measurement.** The measurement process *à la* Everett is described as follows. Bob wishes to perform an elementary measurement on some m-dimensional physical system L in the orthonormal basis $\{|j\rangle_L\}$. Denote by the *reference system* R that subsystem of the universe $\mathcal{U}$ which is entangled with L, so that RL is in a pure state

$$|\Psi\rangle_{RL} = \sum_{j=1}^{m} \lambda_j |\phi_j\rangle_R |j\rangle_L.$$

The *unobserved* measurement consists of the measurement apparatus M, initially in some pure state, becoming entangled with RL via some unitary operator acting on LM only. The state of RLM becomes

$$|\Psi\rangle_{RLM} = \sum_{j=1}^{m} \lambda_j |\phi_j\rangle_R |j\rangle_L |j\rangle_M,$$

where $\{|j\rangle_M\}$ is an orthonormal basis for M. The *observed* measurement consists of the production of an m-dimensional system $B \in \mathcal{K}$ in some pure state, followed by a unitary transformation acting on MB only. This results in

$$|\Psi\rangle_{RLMB} = \sum_{j=1}^{m} \lambda_j |\phi_j\rangle_R |j\rangle_L |j\rangle_M |j\rangle_B,$$

where $|j\rangle_B$ form an orthonormal basis for the Hilbert space $\mathcal{H}_B$ of B. These should be thought of as shorthand notation for the mutually exclusive states of Bob's knowledge with respect to the observation of M, $|j\rangle_B \equiv |\text{observe } M \text{ in the state } |j\rangle_M\rangle_B$. The density matrix ρ_B of B, viewed in the $|j\rangle_B$ basis, has diagonal elements $|\lambda_j|^2$. We define the associated random variable $\mathbf{B}$ as

$$Pr(\mathbf{B} = j) = \langle j|\rho_B|j\rangle_B = |\lambda_j|^2.$$

The Shannon entropy of $\mathbf{B}$ is defined as $H(\mathbf{B}) = -\sum_j Pr(\mathbf{B} = j) \log Pr(\mathbf{B} = j)$. The crucial point is the following. We started off by modelling the mutually exclusive states of Bob's knowledge by an orthonormal basis for B. If ρ_B were diagonal in this basis, it would have the natural interpretation of B being in the state $|j\rangle_B$ with probability $Pr(\mathbf{B} = j)$, and thus isomorphic to the random variable $\mathbf{B}$. However, in general off-diagonal elements do exist; the theory then postulates that Bob is blind to this fact, since it is not in accord with his classical probabilistic vision, referred to henceforth as "diagonal vision." This is, of course, just a manner of speech; all that is being said is that only the diagonal elements have an intepretation.

One might think that if Bob's experience is not based in "reality" there should be a discrepancy between his experience and that of others. This is not the case. If his friend Charlie takes a look at the readout of M, a new system C will be produced in such a way that the joint system $RLMBC$ is now in the state

$$|\Psi\rangle_{RLMBC} = \sum_{j=1}^{m} \lambda_j |\phi_j\rangle_R |j\rangle_L |j\rangle_M |j\rangle_B |j\rangle_C,$$

with the $|j\rangle_C$ defined analogously to $|j\rangle_B$. To compare Bob's and Charlie's experiences we restrict attention to the diagonal elements of the *joint* density matrix ρ_{BC} associated with the joint random variable $\mathbf{BC}$. It can be easily verified that $\mathbf{B}$ and $\mathbf{C}$ are perfectly correlated, namely $Pr(\mathbf{B} = \mathbf{C}) = 1$. In information theoretical terms, we have

$$I(\mathbf{B};\mathbf{C}) = H(\mathbf{B}) = H(\mathbf{C}),$$

where $I(\mathbf{B};\mathbf{C}) = H(\mathbf{B}) + H(\mathbf{C}) - H(\mathbf{BC})$ is the mutual information between $\mathbf{B}$ and $\mathbf{C}$. The same happens when Charlie does not observe the readout of M, but instead measures L in the same basis $\{|j\rangle_L\}$ with his own apparatus N. Bob can also perform a *generalized* measurement on some physical system Q. This is done by entangling Q with L via some unitary operator acting on QL, and subsequently performing an elementary measurement on L, as described above.

(ii) **Preparation.** Alice wishes to prepare some $Q \in \mathcal{P}$ according to an *ensemble* of pure states $\{(p_i, |\psi_i\rangle_Q) : i = 1, \ldots n\}$, $\sum_{i=1}^n p_i = 1$, by which we mean that the state $|\psi_i\rangle_Q$ is prepared with probability p_i. This can be accomplished by

1) measuring Q and bringing it into some fixed state $|0\rangle_Q$, thereby disentangling it from its original reference system.

2) performing conditional unitary operations according to the ensemble $\{(p_i, U_i) : i = 1, \ldots n\}$, such that $|\psi_i\rangle_Q = U_i|0\rangle_Q$.

We omit the description of process 1); suffice it to say that the end result is the system A_1Q being in the state

$$|\Psi\rangle_{A_1Q} = |0\rangle_{A_1}|0\rangle_Q,$$

where $A_1 \in \mathcal{K}_{Alice}$ and $|0\rangle_{A_1}$ represents Alice knowing that Q is in the state $|0\rangle_Q$. Now the outcome of process 2) is the creation of $A_2 \in \mathcal{K}_{Alice}$ so that the state of A_1QA_2 is

$$|\Psi\rangle_{A_1QA_2} = |0\rangle_{A_1} \sum_{i=1}^{n} \sqrt{p_i}|\psi_i\rangle_Q|i\rangle_{A_2}, \quad (3)$$

where the basis state $|i\rangle_{A_2}$ stands for Alice knowing that she has performed the unitary operation U_i. Finally $A \in \mathcal{K}_{Alice}$ is produced giving rise to

$$|\Psi\rangle_{A_1QA_2A} = |0\rangle_{A_1} \sum_{i=1}^{n} \sqrt{p_i}|\psi_i\rangle_Q|i\rangle_{A_2}|i\rangle_A, \quad (4)$$

where the basis state $|i\rangle_A$ represents Alice knowing that Q is in the state $|i\rangle_Q$ *after* the application of the conditional unitary transformation. The transition from (3) to (4) resembles a primitive quantum computation [4].

Upon preparation, the system Q is in the mixed state $\rho_Q = \sum_{i=1}^n p_i|\psi_i\rangle\langle\psi_i|_Q$. The density matrix ρ_A of A, viewed in the $|i\rangle_A$ basis, has diagonal elements p_i. Just as with Bob, these diagonal elements represent the probabilities of Alice experiencing the corresponding basis states and we associated them with the random variable $\mathbf{A}$.

(iii) **Communication.** Communication from Alice to Bob takes place by Alice encoding a message by preparing a physical system Q and Bob subsequently performing a generalized measurement on it. Thus the procedures of (i) and (ii) are combined, with the composite system QA_2A now playing the role of the reference system R. Consequently A and B become entangled via A_2QLM. The diagonal elements of their joint density matrix ρ_{AB} are now associated with the joint random variable $\mathbf{AB}$. The mutual information between what Alice prepared and what Bob received is simply $I(\mathbf{A};\mathbf{B})$. Thus it can be read off very simply from the joint density matrix of Alice and Bob.

Directed entanglement and quantum channels. The theory hitherto presented suggests that all communication can be viewed as entanglement transfer. Initially the sender A is entangled with A_2Q only. Gradually the entanglement is passed on through L, M and finally to B, the receiver. Intuition suggests that the final entanglement between A and B cannot exceed the initial entanglement between A and Q. In addition, one would expect Alice's and Bob's diagonal vision to further reduce their "experienced" entanglement. We now make these ideas concrete by introducing the notion of *directed entanglement*.

Directed entanglement $E(X \rightarrow Y)$ from the system X to the system Y is defined as

$$E(X \rightarrow Y) = S(Y) - S(XY), \tag{5}$$

where $S(Y)$ is short for the von Neumann entropy $S(\rho_Y)$ of the density matrix ρ_Y, $S(\rho_Y) = -tr(\rho_Y \, log \, \rho_Y)$. $S(XY)$ is defined analogously. $E(X \rightarrow Y)$ is readily seen to be the negative of the conditional entropy $S(X|Y) = S(XY) - S(Y)$, a quantity investigated in some detail in [4]. We list some useful properties of $E(X \rightarrow Y)$:

(a) $-S(X) \leq E(X \rightarrow Y) \leq S(X)$

(b) $E(X \rightarrow Y) \leq E(X \rightarrow YZ)$

(c) $E(XY \rightarrow Z) = E(X \rightarrow Z) + E(Y \rightarrow XZ)$

(d) $E(X \rightarrow YZ) \geq E(X \rightarrow Y) + E(X \rightarrow Z)$

(e) $E(XY \rightarrow ZW) \geq E(X \rightarrow Z) + E(Y \rightarrow W)$

(f) $E(X \rightarrow Y) \geq E(X \rightarrow Y^c) \geq E(X^c \rightarrow Y^c)$

(g) $E(X \rightarrow Y)$ is invariant under local unitary transformations $U_X \otimes U_Y$

(h) $-S(X) \leq E(X \rightarrow Y^c) \leq 0$

We omit the proofs, many of which can be found in [4]. In (f) and (h), X^c refers to the *classicized* system X, stripped of its off-diagonal elements in some preferred basis. Note that $S(X^c) = H(\mathbf{X})$. We now use these properties to prove the two main theorems of classical and quantum information processing: the *Holevo bound* and the *quantum data processing inequality*.

Consider sending classical information over some noisy channel $\mathcal{E} : \rho_Q \rightarrow \mathcal{E}(\rho_Q)$. This differs from (iii) in that the system Q gets entangled with some unobserved environment E (via some unitary transformation U_{QE}) between the preparation and measurement phases. The total system involved in the process is thus AA_2QELMB. Just after the interaction with E the system AQ is described by the density matrix

$$\rho_{AQ} = \sum_{i=1}^{n} \sqrt{p_i} |i\rangle\langle i|_A \mathcal{E}(|\psi_i\rangle\langle\psi_i|_Q),$$

and hence

$$E(A \rightarrow Q) = \chi - H(\mathbf{A}),$$

where the *Holevo quantity* χ is given by

$$\chi = S(\mathcal{E}(\rho_Q)) - \sum_{i=1}^{n} p_i S(\mathcal{E}(|\psi_i\rangle\langle\psi_i|_Q)).$$

Denoting by primes quantities calculated after the interaction with LMB, we have the following string of equalities and inequalities

$$E(A \rightarrow Q) = E(A \rightarrow QLMB) = E'(A \rightarrow QLMB)$$

$$\geq E'(A \to B) \geq E'(A^c \to B^c) = H(\mathbf{B}) - H(\mathbf{AB}).$$

The first four relations are due to (b′), (g), (b), and (f) respectively. This gives rise to the *Holevo bound*

$$I(\mathbf{A};\mathbf{B}) \leq \chi. \tag{6}$$

Equality is asymptotically achieved by block coding in the limit of large blocklength [5].

One can also send *quantum* information over a noisy channel. Consider the physical system Q being sent through two noisy channels $\mathcal{E}_1$ and $\mathcal{E}_2$ consecutively. Thus Q, initially entangled with the reference system R only, gets entangled first with E_1 and then with E_2 via some U_{AE_1} and U_{AE_2}, respectively. We denote by primes quantities calculated after the interaction with environment E_1 and by double primes those calculated after the interaction with E_2. Then we have, by (b),

$$E''(R \to QE_1E_2) \geq E''(R \to QE_2) \geq E''(R \to Q).$$

Noting $E''(R \to QE_1E_2) = E(R \to QE_1E_2) = E(R \to Q)$ and $E''(R \to QE_2) = E'(R \to QE_2) = E'(R \to Q)$, both consequences of (g) and (b′), we get the quantum data processing inequality

$$S(\rho_Q) \geq I_c(\rho_Q, \mathcal{E}_1) \geq I_c(\rho_Q, \mathcal{E}_2 \circ \mathcal{E}_1), \tag{7}$$

where I_c is the coherent information [4]. Equality is again achieved asymptotically [6].

We have seen that the key quantum noisy channel relations for sending classical (6) and quantum (7) information both follow from the properties of directed entanglement. It should be stressed that the proofs given are mathematically equivalent to already existing ones [4], only reexpressed in the common language of entanglement transfer.

Quantum thermodynamic entropy. The second law of thermodynamics states that the probability of observing a decrease in the thermodynamic entropy of a closed system tends to zero in the thermodynamic limit. If thermodynamic entropy is interpreted as the von Neumann entropy, then it actually stays *constant* for a closed system, due to the invariance of the von Neumann entropy under local unitary operations; consequently thermal equilibrium can never be reached. With the developments of the previous sections it becomes clear that the *thermodynamic entropy can only be defined relative to some observer.* We define the thermodynamic entropy $S_T(Q|B)$ of a system Q relative to $B \in \mathcal{K}_{Bob}$ as

$$S_T(Q|B) = -E(Q \to B^c), \tag{8}$$

i.e. it is the negative of the directed entanglement from Q to the classicized system B. This is satisfying since, as we have seen, it is through entanglement that Bob is able to receive information, and thermodynamic entropy is thought of as lack of information. By property (h), it lies between 0 and $S(Q)$, the upper bound attained when B is unentangled with Q. By property (g) $E(Q \to B)$ is invariant under local operations on Q, just like $S(Q)$. However, $E(Q \to B^c)$ may well change in either direction. If initially ρ_B is diagonal in the preferred basis, which happens, e.g., if Bob has just performed an elementary measurement on Q, then $S_T(Q|B) = 0$ and it can only increase under local operations on Q. However, by reversing the operation, $S_T(Q|B)$ decreases back to 0. This confirms our understanding that the second law is a merely asymptotic result [8]. Now it should be feasible to rigorously prove the approach to thermal equilibrium with the temperature determined by the Hamiltonian governing the evolution and the initial average energy. Once the initial information is "washed out" all observers will agree on the entropy of the system. For proving the zeroth law it suffices that the sum of thermodynamic entropies of two initially unentangled systems cannot decrease when they interact (see [7] for a full treatment). This follows from properties (b) and (c), which give

$$S_T(Q_1|B) + S_T(Q_2|B) \geq S_T(Q_1Q_2|B),$$

and noting that equality holds when Q_1 and Q_2 are unentangled. We have thus resolved the paradox of constant thermodynamic entropy in closed quantum systems by way of properly defining it in (8).

As for Maxwell's demon, he can indeed decrease $S_T(Q|B)$ by sending Bob classical information about Q. In order for the demon to posess such information he must be entangled with Q. Bob subsequently gets entangled with Q *via* the demon. The system Q is no longer closed, so there is no violation of the second law.

As Everett [2] points out, physical theories have always consisted of two parts, the "formal" mathematical part and the "interpretive" part: a set of rules for connecting the mathematical formalism to our conceptual experience. Unfortunately, the unrigorous nature of the second part often leads to a successful model being mistakenly identified with "reality." We avoid this pitfall by including the conceptual experience in the model itself. The result of such an unconventional approach is a drastically simplified and unified view of physics. The model presented allows for the unification of quantum and classical information processing, quantum mechanics and thermodynamics and non-relativistic physics and Shannon's information theory. It remains to investigate its full implications.

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