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Summary: Professor Cover's group has focused on information theory and the duality of the theory of data compression and the theory of channel capacity [1,2], both classically and quantum mechanically. The search for duality leads to many insights where the results from one domain become the results in another [3,4,5]. Open problems include the amount of information that can be stored in a quantum oracle, as well as an investigation of the quantum multiple access channel [6] and the characterization of the quantum Slepian-Wolf theorem.

Information theory and quantum information (Jon Yard): Statistical thermodynamics introduced a mathematical interpretation of the entropy of a system, as well as a means of interpreting average responses of a bulk statistical mixture to external degrees of freedom. This idea was continued in two distinct directions. The first of these was the development of quantum statistical mechanics, which merged considerations of the quantum behaviour of individual particles with the statistics of bulk systems. The second direction was based on the idea that information should be interpreted as an abstract entity in and of itself. This track is due to Shannon, whose intuition led the way for what has become the field of information theory. Quantum information theory ultimately represents an effort to merge these two directions on both the large and small scales.

There has been initial progress towards generalizing most of the results of classical information theory. Most of this work benefits from the correspondence between classical probability distributions and the eigenvalues of quantum density matrices. Shumacher's coding theorem, which asserts that the Von Neumann entropy $S(\rho) = -\text{tr}\rho \log \rho$ of the density matrix corresponding to an ensemble of pure states represents a minimum average dimension of Hilbert space required to "faithfully" transmit or store a sequence of identically prepared physical systems, takes advantage of this correspondence. Indeed, if the eigenvalues of ρ are $\{\lambda_i\}$, then $S(\rho) = H(\{\lambda_i\})$.

Our notion of "faithful" above can be made more precise, but this also asks us to rethink our notion of ensemble. In one sense, we may consider a probability distribution over the set of all pure states of a prototypical system, then consider preparing a sequence of identical systems, each in a state which corresponds to a (classical) random variable. The density matrix can be thought of as the expected prepared state, where each pure state corresponds to a projection operator. We may now define the fidelity to be the average probability that a system will pass a test which checks to see if the state has changed during a physical process.

We may also take a different viewpoint towards ensembles as arising from entanglement with some external system. Theoretically, such a system is abstracted as a "purifier." There is a sense in which the purifier can be used to generate any ensemble which respects the partial density matrix it induces. This leads to an interpretation of the purifier as an oracle. Entangled ensembles come with a "goodness criterion" as well, the entanglement fidelity F_e , which reflects how well entanglement with an oracle is maintained.

Shumacher compression gives the same rates for both of the above types of ensembles. However, this is not the case when multi-partite ensembles are considered. Here, the analogy between probabilities and eigenvalues of ρ breaks down. Heuristically, this can be understood by the fact that the density matrix of a bipartite system can be thought of as a "matrix of matrices." Since a general ring of hermitian matrices is not guaranteed to be commutative, the naive spectral method fails to apply to all but the simplest multiuser problems.

There is an area of mathematics which falls under the vague name of "quantum groups" (group is a misnomer), which were introduced in order to study noncommutative spaces. The theory can be roughly understood as the application of formal power series to the representation theory of certain Lie algebras, such as Hopf algebras, whose self-dual nature shows potential for putting information-theoretic dualities on a rigorous mathematical and physical footing. Renormalization theory utilizes Hopf algebras to encode the

logical structure of divergences in Feynman diagrams, allowing for computation of perturbative results to very high order. The same mathematical framework has applications in low dimensional topology, the theory of braids, exactly solvable models in statistical mechanics, conformal field theory and infinite dimensional algebras. Hopf algebras have properties which suggest that they may also be a rich language for expressing relationships between nodes in a quantum network.

There has been recent work in the literature concerning the foundations of quantum mechanics suggesting a unified theory of quantization and dequantization; i.e. a model for which many classical limits of a quantum theory may exist (one of which is equivalent to $\hbar \rightarrow 0$), each of which gives the same quantum theory after (re)quantization. This theory naturally adds a symplectic form to a real infinite dimensional Hilbert space via introduction of an antihermitian operator called a complex structure, which is basically a controlled way of turning an infinite dimensional real space to an imaginary one. Operators on the Hilbert space are then associated with real valued functions on a manifold. The complex structure enables us to probe certain parameters of entanglement, such as the concurrence, which is in a sense the determinant of a pure tensor state, viewed as a matrix. It can be shown that no Hermitian observable can compute the concurrence. A symmetric product, or metric, is still retained on the underlying Hilbert space, allowing for a unified framework of linear (unitary propagation, superposition, Hamiltonian flow), and nonlinear (state reduction, decoherence, measurement, Lorentzian geometry) concepts. This shows great promise for a rigorous unification of many heuristic assumptions and analogies to classical physics which have driven the development of quantum theory.

The quaternions \mathbf{H} and the octonions \mathbf{O} appear favorable for describing entangled systems. While the octonions are not associative, they can be represented in terms of a “twisted group algebra,” with connections to conformal field theory as well as to Hopf algebras. Projective geometry over \mathbf{H} and \mathbf{O} also comes into play, as the projective plane \mathbf{OP}^2 represents a topologically rich space containing symmetries which give rise to a relationship between 3 spaces known as a triality. A common example of a triality in physics is that between the representations of a left handed spinor, a right handed spinor and a vector boson, such as a vertex in the standard QED model of an electron interacting with a photon.

Triality shows potential for use in quantum, as well as classical information theory, in that to each triality is associated 3 dualities, each formed by fixing one of the three subsystems. A possible candidate for a triality in quantum information theory consists of the triple $\mathcal{H}_O, \mathcal{H}_E, \mathcal{H}_Q$, corresponding to a cold benevolent controlling oracle, a hot malicious environment, and the quantum system under study. The oracle’s controlled interaction with the quantum system could be used to model, say, the interaction of a controller for a dynamic error correction scheme with the matter in the register of a quantum computer.

The mathematical structures which give rise to and encode the structure of such maps have a well-defined theory. Division algebras, quaternions, octonions, Jordan algebras, projective geometry and the Hopf fibrations of the spheres S^{15}, S^7, S^3 are all useful concepts for analyzing the topology of low dimensional entangled systems. These concepts exist throughout mathematical physics in areas including string theory, M theory, nonabelian gauge theory and condensed matter physics. For instance, the second Hopf map $S^7 \rightarrow S^4$ from the projective quaternionic line to the space of unit vectors in \mathbf{R}^5 was recently used to generalize the 2D quantum Hall effect to 3D edge states on the boundary of a hypothetical 4D fluid of electrons. The *same* map is useful in describing the structure of the set of pure states of a *single* pair of spin- $\frac{1}{2}$ particles.

Dualities have been an underlying theme of most of the work produced by this group. Our vision is to eventually be able to incorporate physical dualities into all of our group’s work, with the hopes of understanding the physical theory of information.

References

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