In quantum mechanics, two particles are called **entangled** if their state cannot be factored into single-particle states:

\[ |\Psi_{12}\rangle \neq |\Psi_1\rangle \otimes |\Psi_2\rangle \]

Measurements performed on the first particle gives reliable information about the state of the second particle, no matter how far apart they may be.

This is the standard **Copenhagen interpretation** of quantum measurements which suggests **nonlocality** of the measuring process.

In difference with classical correlations, quantum correlations do not depend on the basis. Changing the basis does not change the state of quantum entanglement.
The idea of entanglement was introduced into physics by Einstein-Podolsky-Rosen [Phys. Rev., 47, 777 (1935)]

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, Institute for Advanced Study, Princeton, New Jersey
(Received March 25, 1935)

Any serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

Whatever the meaning assigned to the term complete, the following requirement for a complete theory seems to be a necessary one: every element of the physical reality must have a counterpart in the physical theory. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot
According to EPR who consider system of two separate particles:

- A measurement performed on one quantum particle has a **nonlocal** effect on the physical reality of another distant particle – “spooky action at a distance” (entanglement), in which EPR did not believe;

- Quantum mechanics is incomplete [some extra (“hidden”) variables are needed] - **suggested by EPR.**

Bohr replied to the EPR paradox that in a such quantum state one could not speak about the individual properties of each of the particles, even if they were distant from one another.

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle_1 |\rangle_2 - |\rangle_1 |\rangle_2 - |\rangle_1 |\rangle_2 + |\rangle_1 |\rangle_2 \right)
\]
In the mid-sixties it was realized that the nonlocality of nature is a testable hypothesis.

1964 - Bell’s Inequalities [(Physics, 1, 195 (1964))):

John Bell showed that the “locality hypothesis” with hidden variables leads to a conflict with quantum mechanics.

He proposed a mathematical theorem containing certain inequalities. An experimental violation of his inequalities would suggest the states obeying the quantum mechanics with nonlocality.

\[
I = \left| \sqrt{I_1} e^{i\phi} + \sqrt{I_2} e^{i\theta} \right|^2 = I_1 + I_2 + 2\cos(\phi - \theta)\sqrt{I_1 I_2}
\]

Subsequent experimental realization of EPRB “gedanken experiment” confirmed the quantum predictions (Bell’s inequalities were violated):

Creation of Polarization Entangled Photons: Spontaneous Parametric Down Conversion

Type I BBO crystals

\[ |\Psi\rangle_{SPDC} = |V\rangle_s |V\rangle_i + e^{i\phi} |H\rangle_s |H\rangle_i \]

With phase \( \phi \) equal 0 or \( \pi \) (which can be compensated using a quartz plate) and with a normalization

\[ |\Psi_{EPR}\rangle = \frac{1}{\sqrt{2}} \left( |V\rangle_s |V\rangle_i \pm |H\rangle_s |H\rangle_i \right) \]

In this state H-polarized SPDC light cone overlaps V-polarized light cone, so SPDC light in this state is not polarized.
If we measure the polarizations of signal and idler photons in the $H, V$ basis, there are two possible outcomes: both vertical and both horizontal. We could instead measure the polarizations with polarizers rotated by an angle $\alpha$, which is arbitrary. We use the rotated polarization basis

$$|V_\alpha\rangle = \cos \alpha |V\rangle + \sin \alpha |H\rangle \quad \text{and} \quad |H_\alpha\rangle = -\sin \alpha |V\rangle + \cos \alpha |H\rangle$$

In this basis the state is

$$|\Psi_{Bell}\rangle = \frac{1}{\sqrt{2}} (|V\rangle_s |V\rangle_i + |H\rangle_s |H\rangle_i)$$

So if we measure in this rotated basis, we obtain the same results, because the state is rotation invariant.

By placing polarizers rotated to arbitrary angles $\alpha$ and $\beta$ in the idler and signal paths, we measure the polarization of the downconverted photons.

Measured coincidence count

$$N(\alpha, \beta) \sim P_{VV}(\alpha, \beta)$$

where

$$P_{VV}(\alpha, \beta) = \left| \left< V_\alpha | V_\beta \right| \Psi_{Bell} \right|^2$$

and $VV$ subscripts on $P$ indicate the measurement outcome $V_\alpha V_\beta$, both photons vertical in the bases of their respective polarizers.

It is easy to show that for the EPR state

$$P_{VV}(\alpha, \beta) = \frac{1}{2} \cos^2 (\beta - \alpha)$$
More generally, there are four possible outcomes:

\[
P_{VV}(\alpha, \beta) = \frac{1}{2} \cos^2(\beta - \alpha) \quad P_{HH}(\alpha, \beta) = \frac{1}{2} \cos^2(\beta - \alpha)
\]
\[
P_{VH}(\alpha, \beta) = \frac{1}{2} \sin^2(\beta - \alpha) \quad P_{HV}(\alpha, \beta) = \frac{1}{2} \sin^2(\beta - \alpha)
\]

We can find this probabilities in the experiment by measuring coincidence count \(N(\alpha, \beta)\)

\[
P_{VV}(\alpha, \beta) = \frac{N(\alpha, \beta)}{N_{total}} = \frac{N(\alpha, \beta)}{N(\alpha, \beta) + N(\alpha_{\perp}, \beta_{\perp}) + N(\alpha, \beta_{\perp}) + N(\alpha_{\perp}, \beta)}
\]
\[
P_{HH}(\alpha, \beta) = \frac{N(\alpha_{\perp}, \beta_{\perp})}{N(\alpha, \beta) + N(\alpha_{\perp}, \beta_{\perp}) + N(\alpha, \beta_{\perp}) + N(\alpha_{\perp}, \beta)}
\]
\[
P_{VH}(\alpha, \beta) = \frac{N(\alpha, \beta_{\perp})}{N(\alpha, \beta) + N(\alpha_{\perp}, \beta_{\perp}) + N(\alpha, \beta_{\perp}) + N(\alpha_{\perp}, \beta)}
\]
\[
P_{HV}(\alpha, \beta) = \frac{N(\alpha_{\perp}, \beta)}{N(\alpha, \beta) + N(\alpha_{\perp}, \beta_{\perp}) + N(\alpha, \beta_{\perp}) + N(\alpha_{\perp}, \beta)}
\]

where \(N_{total}\) is the total number of pairs detected and \(\alpha_{\perp}\) and \(\beta_{\perp}\) are the polarizer settings \(\alpha + 90^\circ, \beta + 90^\circ\).

Let us introduce \(E(\alpha, \beta) \equiv P_{VV} + P_{HH} - P_{VH} - P_{HV}\)

It is easy to show that \(E(\alpha, \beta) = \cos 2(\alpha - \beta)\)

For completely correlated events \(E(\alpha = \beta) = 1\) and for complete anticorrelation \(E(\alpha - \beta = \pi / 2) = -1\)
Calculation of Bell Inequality


Bell’s inequality defines the degree of polarization correlation under measurements at different polarizer angles. The proof involves two measures of correlations:

\[ E(\alpha, \beta) = P_{VV}(\alpha, \beta) + P_{HH}(\alpha, \beta) - P_{VH}(\alpha, \beta) - P_{HV}(\alpha, \beta), \]

and

\[ S = |E(a,b) - E(a,b')| + |E(a',b) + E(a',b')|, \]

where

\[ E(\alpha, \beta) = \frac{N(\alpha, \beta) + N(\alpha_\perp, \beta_\perp) - N(\alpha, \beta_\perp) - N(\alpha_\perp, \beta)}{N(\alpha, \beta) + N(\alpha_\perp, \beta_\perp) + N(\alpha, \beta_\perp) + N(\alpha_\perp, \beta)} \]

The above calculation of \( S \) requires a total of 16 coincidence measurements (\( N \)), at polarization angles \( \alpha \) and \( \beta \):

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( b = -22.5^\circ )</th>
<th>( b' = 22.5^\circ )</th>
<th>( b_\perp = 67.5^\circ )</th>
<th>( b'_\perp = 112.5^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = -45^\circ )</td>
<td>162</td>
<td>41</td>
<td>93</td>
<td>204</td>
<td></td>
</tr>
<tr>
<td>( a' = 0^\circ )</td>
<td>154</td>
<td>186</td>
<td>51</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>( a_\perp = 45^\circ )</td>
<td>36</td>
<td>183</td>
<td>223</td>
<td>102</td>
<td></td>
</tr>
<tr>
<td>( a'_\perp = 90^\circ )</td>
<td>56</td>
<td>57</td>
<td>278</td>
<td>267</td>
<td></td>
</tr>
</tbody>
</table>

Violation of Bell inequality if \( |S| > 2 \)
It is very important to know that Bell inequality is violated only for some angles $\alpha$ and $\beta$. For other angles both quantum theory and classical physics (HVT) give the same value of $S$.

$$S = |E(a,b) - E(a',b')| + |E(a',b) + E(a',b')|$$

$$= |\cos 2(a-b) - \cos 2(a - b')| + |\cos 2(a-b') + \cos 2(a' - b')|$$

For the particular choice of polarizing angles $a = -\pi/4$, $a' = 0$, $b = -\pi/8$, $b' = \pi/8$, we obtain the result

$$\left| - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right| + \left| \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right| = 2\sqrt{2} \approx 2.828 \quad S_{\text{max}} = 2\sqrt{2}$$

Comparison of HVT (dashed line) and quantum theory (solid line)

If in our experiment we measure $N(\alpha, \beta)$ with fringe visibility $E(\alpha, \beta) = V \cos 2(\alpha - \beta)$ and $S_{\text{max}} = 2\sqrt{2V}$,

$$V = \frac{N_{\text{max}} - N_{\text{min}}}{N_{\text{max}} + N_{\text{min}}}$$

so if we have $V \leq \frac{1}{\sqrt{2}} = 0.71$,

Bell inequality will not be violated at any angles.

Quantum correlation (entanglement) is a very rare event and it is difficult to observe !!!
Proof of CHSH Bell inequality

\[ |E(\alpha, \beta) - E(\alpha, \beta')| + |E(\alpha', \beta) + E(\alpha', \beta')| \leq 2 \]

This proof is based on a trivial mathematical relation

\[ |a + b + c + ...| \leq |a| + |b| + |c| + ... \]

In a classical case, the distribution of the hidden variable \( \lambda \) is described by a function \( \rho(\lambda) \),

where \( \rho(\lambda) \geq 0 \) \quad \int \rho(\lambda)d\lambda = 1 \]

The assumption of locality and realism are embodied in the assumption that for the idler photon the outcome of measurement is determined completely by \( \lambda \) and the measurement angle \( \alpha \), and for the signal photon – by \( \lambda \) and \( \beta \).

These outcomes are specified by the functions \( A(\lambda, \alpha) = \pm 1 \) for detection as \( V_\alpha \) and \( H_\alpha \), and

\( B(\lambda, \beta) = \pm 1 \) for detection as \( V_\beta \) and \( H_\beta \).

Probabilities of particular outcomes are given by the integrals

\[ P_{vv}(\alpha, \beta) = \int \frac{1 + A(\lambda, \alpha)}{2} \frac{1 + B(\lambda, \beta)}{2} \rho(\lambda)d\lambda \]

\[ P_{hh}(\alpha, \beta) = \int \frac{1 - A(\lambda, \alpha)}{2} \frac{1 - B(\lambda, \beta)}{2} \rho(\lambda)d\lambda \]

\[ P_{vh}(\alpha, \beta) = \int \frac{1 + A(\lambda, \alpha)}{2} \frac{1 - B(\lambda, \beta)}{2} \rho(\lambda)d\lambda \]

\[ P_{hv}(\alpha, \beta) = \int \frac{1 - A(\lambda, \alpha)}{2} \frac{1 + B(\lambda, \beta)}{2} \rho(\lambda)d\lambda \]

It is easy to show that

\[ E(\alpha, \beta) \equiv P_{vv} + P_{hh} - P_{vh} - P_{hv} = \int A(\alpha, \lambda)B(\beta, \lambda)\rho(\lambda)d\lambda \]
\[ |a + b + c + ...| \leq |a| + |b| + |c| + ... \]

\[
|E(\alpha, \beta) - E(\alpha, \beta')| = \left| \int \left[ A(\alpha, \lambda)B(\beta, \lambda) - A(\alpha, \lambda)B(\beta', \lambda) \right] \rho(\lambda) d\lambda \right| \\
\leq \int |A(\alpha, \lambda)B(\beta, \lambda) - A(\alpha, \lambda)B(\beta', \lambda)| \rho(\lambda) d\lambda = \int |A(\alpha, \lambda)[B(\beta, \lambda) - B(\beta', \lambda)]| \rho(\lambda) d\lambda \\
= \int |B(\beta, \lambda) - B(\beta', \lambda)| \rho(\lambda) d\lambda
\]

We used relation \[ |A(\alpha, \lambda)| = 1 \]

In the same manner \[ |E(\alpha', \beta) + E(\alpha', \beta')| \leq \int |B(\beta, \lambda) + B(\beta', \lambda)| \rho(\lambda) d\lambda \]

\[
|E(\alpha, \beta) - E(\alpha, \beta')| + |E(\alpha', \beta) + E(\alpha', \beta')| \leq \int \left[ |B(\beta, \lambda) - B(\beta', \lambda)| + |B(\beta, \lambda) + B(\beta', \lambda)| \right] \rho(\lambda) d\lambda
\]

Because \[ B = \pm 1 \]
\[
|B(\beta, \lambda) - B(\beta', \lambda)| + |B(\beta, \lambda) + B(\beta', \lambda)| = 2
\]


\[
|E(\alpha, \beta) - E(\alpha, \beta')| + |E(\alpha', \beta) + E(\alpha', \beta')| \leq 2
\]
Recent advances in quantum communication

Quantum communication in Space

QUEST = QUantum Entanglement in Space Experiments (ESA)

For the proposed International Space Station experiment, the entangled photons would be beamed from orbit to two distant ground stations, allowing them to communicate using the quantum key.