Determination of modes of elliptical waveguides with ellipse transformation perturbation theory

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All optical fibers, including single-mode, multimode, and multicore fibers, exhibit some degree of birefringence, either purposely, such as in polarization-maintaining fibers, or inadvertently due to material or fabrication imperfections. Finding a low-complexity method to accurately calculate the modal characteristics of elliptical fibers has been a long-standing problem. We present a novel accurate perturbative method that avoids the difficulties associated with the traditional Mathieu function treatment. The method is also applicable to a broader class of oscillating systems with elliptical geometry. © 2017 Optical Society of America

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The accurate determination of transverse modes of elliptical waveguides is important for both deliberately elliptical waveguides, such as polarization-maintaining fibers, but also in the context of circular fibers due to core shape fluctuations caused by manufacturing imperfections [1]. Recently, multicore fibers with elliptical cores have been shown to be promising candidates for next-generation space-division multiplexing applications in optical communication networks [2]. Traditionally, elliptical waveguides have been treated using the elliptical coordinate system that invariably leads to Mathieu differential equations [3]. The normal modes are then expressed as linear combinations of products of angular and radial Mathieu functions, both of which are notoriously difficult to compute numerically [4,5]. The angular Mathieu functions are expressed as Fourier series and the radial functions as series of products of Bessel functions, and the coefficients for the terms are obtained by calculating the eigenvectors of an infinite matrix.

The determination of transverse modes of step-index waveguides requires the matching of tangential fields at the core–cladding interface. For circular waveguides, this is quite trivial and boils down to matching an oscillating Bessel function \( J_n \) in the core with a decaying modified Bessel function \( K_n \) in the cladding to ensure continuity of the tangential fields. For elliptical waveguides, this procedure is significantly more cumbersome as it requires the matching of an infinite series of products of Mathieu functions in the core with an infinite series of products of different Mathieu functions in the cladding. For this reason, numerous alternative approaches have been suggested. Schneider and Marquardt [6] used a power series to compute Mathieu functions, but the series becomes slowly converging for near-circular geometry. Nor does this approach avoid the tedious matching of two infinite series. Other methods [7–15] rely on heavy approximations and assumptions, such as weak guidance, and are therefore incapable of providing arbitrarily accurate results.

In this Letter we outline a method, dubbed the ellipse transformation perturbation method (ETPM) [16], to determine the modes of elliptical waveguides. ETPM is based on treating ellipticity as a perturbation from the circular case and therefore works especially well for near-circular geometry. The method is significantly simpler than the Mathieu function treatment, more versatile than previous approximate methods, and even in its coarsest form equally or more accurate than any past approximate method. Unlike previous approximate methods, ETPM is not restricted to small relative refractive index difference between the core and the cladding. Furthermore, ETPM is generally applicable to any oscillating physical system with elliptical geometry [17–20].

The refractive index profile of a waveguide with an elliptical cross section oriented along the \( z \) axis can be written as

\[ n(x,y) = \begin{cases} 
  n_1, & (x/a)^2 + (y/b)^2 \leq 1 \\
  n_2, & (x/a)^2 + (y/b)^2 > 1 
\end{cases} \quad (1) \]

Now, instead of switching to the conventional elliptical cylindrical coordinates [3], we apply ETPM and re-scale the \( x \) axis by defining \( \omega = (b/a)x \). In the new coordinates \((\omega,y,z)\), the elliptical core–cladding interface becomes circular-like as it takes the form \( \omega^2 + y^2 = b^2 \).

To determine the waveguide modes, we solve for the longitudinal electric and magnetic field components \( E_z = F(\omega,y)e^{(\omega-z\omega)} \) and \( H_z = G(\omega,y)e^{(\omega-z\omega)} \). In the new coordinates \((\omega,y,z)\), The Helmholtz equation satisfied by \( E_z \) becomes

\[ [(1 + \delta)\omega^2 + \partial_y^2 + (k^2 - \beta^2)]F(\omega,y) = 0, \quad (2) \]

where we have defined \( \delta = (b/a)^2 - 1 \). The parameter \( \delta \) serves as a measure of the ellipticity of the waveguide core, vanishing for a circular waveguide. Note that \( \delta \) can be negative or positive depending on whether the major axis of the ellipse is parallel to
the $x$ or the $y$ axis, and either orientation is fine as long as $|\delta| < 1$. By treating $\delta$ as a perturbation from the circular case, $F$ (and $G$) can be expanded as

$$F(w, y) = \sum_{n=0}^{\infty} \delta^n F_n(w, y),$$

(3)

without loss of generality. The condition $|\delta| < 1$ is needed for the series to converge. For Eq. (3) to satisfy Eq. (2), the equations have to be satisfied individually for each order of $\delta$. This is analogous to perturbation theory commonly used in quantum mechanics with the key difference being that the perturbation affects the coordinate system used. Substituting Eq. (3) into Eq. (2) yields an infinite set of equations:

$$[\partial_{xx}^2 + \partial_{yy}^2 + (k^2 - \beta^2)]F_n(w, y) = 0,$$

(4)

and

$$[\partial_{xx}^2 + \partial_{yy}^2 + (k^2 - \beta^2)]F_n(w, y) = -\partial_x^2 F_n(w, y),$$

(5)

where Eq. (5) was divided by $\delta^n$ and holds for all $n \geq 1$.

Equation (4) is mathematically identical to the Helmholtz equation for a circular fiber, with the general solution given by

$$F_0 = \sum_{n=0}^{\infty} a_n \sin(n\phi + \phi_0)Z_n(\gamma r),$$

(6)

where $Z_n$ is a Bessel function ($J_n$ in the core region and $K_n$ in the cladding region), $\gamma$ is a constant related to $k$ and $\beta$ (with different values in the core and cladding regions), and $a_n$ and $\phi_0$ are constants found by matching the boundary conditions. We used $w = r \cos \phi$ and $y = r \sin \phi$, but $r$ and $\phi$ are not the circular polar coordinates (a constant $r$ corresponds to an ellipse instead of a circle). We refer to them as quasi-polar coordinates (QPCs). In physical terms, the QPCs are polar coordinates after scaling the $x$ axis by the axis ratio of the ellipse.

The deviations of the field profile $F$ from the circular case are given by the higher order terms $F_n$ obtained by solving Eq. (5), which is a driven Helmholtz equation. However, note that for $\delta \neq 0$ the zeroth-order term $F_0$ is already different from the circular waveguide modes as it has been scaled in the $x$ dimension. The homogeneous part of Eq. (5) is identical to Eq. (4), and its general solution has the exact same form as the solution of Eq. (4) given in Eq. (6). The general form of $F$ in Eq. (3) can thus be written as

$$F = \sum_{n=0}^{\infty} c_n Z_n(\gamma r) \sin(n\phi + \phi_0) + \delta^n F_0^{(p)} (r, \phi),$$

(7)

where $F_0^{(p)} (r, \phi)$ are particular solutions of Eq. (5). Even though an elliptical waveguide is not rotationally symmetric, it exhibits reflection symmetry with respect to both the $x$ and $y$ axes. As a result, $F$ must be an even or odd function with respect to both $x$ and $y$. Therefore, either $\phi_0 = 0$ or $\phi_0 = \pi/2$ for all $n$. Furthermore, $c_n = 0$ for even $n$ or $c_n = 0$ for odd $n$. Since $\sin(x + \pi/2) = \cos(x)$, the $\phi_0 = 0$ case will be referred to as the sine case and $\phi_0 = \pi/2$ as the cosine case. In what follows, the treatment will be restricted to the sine case only. The treatment of the cosine case is identical with sines replaced by cosines.

Equation (7) provides the general solution for the waveguide modes as an infinite series in powers of the perturbation parameter $\delta$. To solve Eq. (5) for the first-order correction ($n = 1$), we note that $F_0$ needs to converge toward the circular-waveguide solution in the limit $\delta \to 0$, say $F_0 = A \sin(n\phi)Z_n(\gamma r)$. This means that for each mode the sum in Eq. (6) consists of only one term. Substituting this simple form of $F_0$ into Eq. (5) then gives the equation for $F_1$, the particular solution of which, after some algebra, is found to be

$$F_1^{(p)} = A \frac{p}{8} [J_{n+1}(pr) - J_{n-1}(pr)] \sin(n \phi)$$

$$- A \frac{p}{16} [J_{n+3}(pr) - J_{n+1}(pr)] \sin((n + 2) \phi)$$

$$- A \frac{p}{16} [J_{n-1}(pr) - J_{n-3}(pr)] \sin((n - 2) \phi),$$

(8)

where $p = \sqrt{(k_0 n_1)^2 - \beta^2}$ in the core, and

$$F_1^{(p)} = A \frac{q r}{8} [K_{n+1}(qr) + K_{n-1}(qr)] \sin(n \phi)$$

$$+ A \frac{q r}{16} [K_{n+3}(qr) + K_{n+1}(qr)] \sin((n + 2) \phi)$$

$$+ A \frac{q r}{16} [K_{n-1}(qr) + K_{n-3}(qr)] \sin((n - 2) \phi),$$

(9)

where $q = \sqrt{\beta^2 - (k_0 n_2)^2}$ in the cladding. The obtained result makes intuitive sense. When a circular-waveguide mode of angular dependence $\sin(n \phi)$ is perturbed by making the core slightly elliptical, the first-order correction comes from the $\sin((n \pm 2) \phi)$ terms as they are the nearest-order non-zero perturbative terms. In addition to these particular solution terms, the sum in Eq. (7) also has terms of the form $Z_{\pm 2}(\gamma r) \sin((n \pm 2) \phi)$ or only the $Z_{\pm 1}(\gamma r) \sin((n \pm 2) \phi)$ term if $n < 2$ since the sum in Eq. (7) starts from $n = 0$ that originate from the homogeneous parts of the equations for $F_n$. Since $F_1$ already has terms of the form $\sin((n \pm 2) \phi)$ from the particular solution, we will interpret the $Z_{\pm 2}(\gamma r) \sin((n \pm 2) \phi)$ terms that exist in Eq. (7) as belonging to $F_1$, and other terms of the form $Z_{\pm 2}(\gamma r) \sin((n \pm 2 m) \phi)$ that are present in the sum as being a part of $F_n$. Even though the function $F_{k}$ affects the particular solution of $F_{k+1}$, it does not matter which homogeneous solution terms are or are not included in $F_{k}$, as the coefficients $c_n$ in Eq. (7) are arbitrary (for the time being) and the particular solutions give only the magnitudes of the correct terms with respect to these coefficients. Without specifying which homogeneous solution terms belong to which order $F_n$, only the zeroth-order term $F_0$ would be uniquely defined, aside from its normalization factor, as it has to coincide with the circular-waveguide mode for $\delta = 0$.

Since circular-waveguide modes have an azimuthal mode order $n$ associated with them, and since each elliptical mode converges toward a unique circular mode in the limit $\delta \to 0$, it is meaningful to assign a mode order to the elliptical modes as well. Roughly speaking, the mode order of an elliptical mode is the index $n$ of the dominant term in the sum in Eq. (7) (except for the TE-like $n = 0$ mode). Now consider an elliptical mode of some order $n$. In general, all terms in Eq. (7) are needed, but for near-circular waveguides we have $|\delta| \ll 1$, and hence the first-order approximation $F \approx F_0 + \delta F_1$ can be expected to yield accurate results, as each successive higher order term is smaller by a factor of $|\delta|$ compared to the previous one. To the first order in the perturbation expansion, the transverse profile $F$ of the electric field's $z$ component is found to be
\[
F \approx x_1 j_0 s_n + x_2 j_{n+2} s_{n+2} + x_3 j_{n-2} s_{n-2} + x_4 j_{n+1} \delta_{n+1,n-1} s_n
\]
\[
+ x_5 j_0 \frac{\delta_{n}^{pr}}{8} (j_{n+1} - j_{n-1}) s_n
\]
\[
- x_6 j_2 \frac{\delta_{n}^{pr}}{16} (j_{n+3} - j_{n+1}) s_{n+2} - x_7 j_1 \delta_{n}^{pr} (j_{n-1} - j_{n-3}) s_{n-2}
\]
(10)
in the core, and
\[
F \approx x_8 k_1 s_n + x_9 k_2 s_{n+2} + x_{10} k_1 s_{n-2}
\]
\[
+ x_{11} \delta_{n}^{pr} (k_{n+1} + k_{n-1}) s_n
\]
\[
+ x_{12} k_2 \frac{\delta_{n}^{pr}}{16} (k_{n+3} + k_{n+1}) s_{n+2} + x_{13} \delta_{n}^{pr} (k_{n-1} + k_{n-3}) s_{n-2}
\]
(11)
in the cladding, where we have defined the short-hand notations \( j_j = j_j(pr) \), \( k_i = K(qr) \), and \( s_j = \sin(j \phi) \), and where \( x_1 \) to \( x_{13} \) are the amplitudes of various terms. As mentioned earlier, we can follow the same procedure for the magnetic field \( H_z \) written in terms of the longitudinal fields \( E \) and \( H \), and will obtain identical terms for \( E_z \).

The equations can be written in matrix form as
death
\[ E_r = i \mu_0 \frac{\nu^2}{k_0^2} \begin{bmatrix} (b \cos^2 \phi + a \sin^2 \phi) \frac{\partial E_z}{\partial r} + \frac{(a - b) \sin \phi \cos \phi \frac{\partial H_z}{\partial \phi}}{r} \\
+ \frac{\omega \mu}{\beta} \frac{(a - b) \sin \phi \cos \phi \frac{\partial H_z}{\partial \phi}}{r} + \frac{\omega \mu a \cos^2 \phi + b \sin^2 \phi \frac{\partial E_z}{\partial \phi}}{r} \\
\end{bmatrix} \hat{E} + \frac{i \beta}{\nu^2} \begin{bmatrix} (b \cos^2 \phi + a \sin^2 \phi) \frac{\partial H_z}{\partial r} + \frac{(a - b) \sin \phi \cos \phi \frac{\partial E_z}{\partial \phi}}{r} \\
+ \frac{\omega \mu}{\beta} \frac{(a - b) \sin \phi \cos \phi \frac{\partial E_z}{\partial \phi}}{r} + \frac{\omega \mu a \cos^2 \phi + b \sin^2 \phi \frac{\partial H_z}{\partial \phi}}{r} \\
\end{bmatrix} \hat{H}, \]
(12)
where \( k_0^2 = p^2 \) in the core and \( k_0^2 = -q^2 \) in the cladding. Even though these expressions appear lengthy, it should be kept in mind that they involve only trigonometric and Bessel functions and are thus considerably simpler than the Mathieu function treatment [3]. It is important to note that the transverse tangential field components are not parallel to the unit vector \( \hat{E} \) in the QPC system. However, the transverse component tangential to the interface simplifies considerably and can be expressed as
\[ E_t = i \mu_0 \frac{\nu^2}{k_0^2} \begin{bmatrix} (b^2 - a^2) \sin(2 \phi) \frac{\partial \phi}{\partial r} E_{\phi} + \frac{\omega \mu}{\beta} \frac{\partial E_{\phi}}{\partial r}
\end{bmatrix} \hat{E}, \]
(14)
and
\[ H_t = -\frac{\nu^2}{k_0^2} \begin{bmatrix} (b^2 - a^2) \sin(2 \phi) \frac{\partial \phi}{\partial r} E_{\phi} - \frac{\omega \mu}{\beta} \frac{\partial E_{\phi}}{\partial r}
\end{bmatrix} \hat{H}, \]
(15)
where \( N = \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi} \). Note that Eqs. (12)-(15) are the same for the cosine case, as the trigonometric functions in them originate from the coordinate system used and not the fields.

We need to match the tangential components of the electric and magnetic fields \( (E_t, H_t) \) at the core-cladding interface. Matching of \( E_t \) at the interface yields three equations in the general case because of the three different \( \phi \) terms in Eqs. (10) and (11). Similarly, matching of \( H_t \) \( E_t \), and \( H_t \) yields nine more equations. The equations can be written in matrix form as \( MV = 0 \), where \( V = [x_1, \ldots, x_n, y_1, \ldots, y_n]^T \). The modal propagation constants \( \beta \) are found by requiring \( \det(M) = 0 \), and the eigenvector of \( M \) corresponding to the zero eigenvalue then gives the 12 coefficients \( x_i \) and \( y_i \), up to a multiplicative factor. For \( n < 3 \), the number of linearly independent equations is smaller. The same procedure can be used to include second- and higher order perturbation terms, but the number of equations grows for higher order terms because of the increasing number of variables \( x_i \) and \( y_i \).

Depending on the effective area of the fundamental mode compared to the core area, even slight ellipticity in the core of an optical fiber can have a significant effect on the birefringence between the slow and fast fundamental modes [3]. Manufacturing imperfections thus cause fibers to be randomly birefringent due to core ellipticity and stress-induced anisotropic changes in the refractive index. This is generally detrimental in optical communication systems and a limiting factor for networks based on optical fibers [1]. Varying core ellipticity may also lead to linear mode-coupling in multimode fibers, which would be problematic for next-generation fiber networks utilizing space-division multiplexing [21]. Accurate modeling of random changes in the effective indices and linear coupling coefficients would require the
The indices of the core and cladding are constants of the slow and fast fundamental modes. The refractive 
\[ \Delta \] 
differs for the two modes, and \( n_1 \) and \( n_2 \) respectively. The fiber core has an axis ratio of 
\( a/b \), with \( a \) being the major axis. The ETPM is the only method, aside from the tedious 
"exact" Mathieu function treatment, capable of yielding arbitrarily accurate results.

In conclusion, we introduced a method, called the ellipse transformation perturbation method (ETPM), to determine 
the normal modes of oscillating systems with elliptical geometry. The method is mathematically considerably simpler than 
the conventional Mathieu function treatment and gives quite accurate results for nearly circular geometries, even with just the first-order 
corrective term, as was demonstrated in the context of birefringence in optical fibers with elliptical cores. Even for a relatively 
high core ellipticity (axis ratio 0.90), the inclusion of first-order corrections yielded results for fiber birefringence that were 
comparable to or more accurate than several previous approximate methods. ETPM avoids assumptions, such as weak guidance, 
and has the additional advantage of being able to provide both the modal eigenvalues and the eigenfunctions (corresponding 
to propagation constants and modal field distributions in the context of optical waveguides) for any mode to any desired accuracy.

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