# DOUBLE RESONANCE SELF-INDUCED TRANSPARENCY

C.R. STROUD Jr. and D.A. CARDIMONA
*The Institute of Optics, University of Rochester, Rochester, New York 14627, USA*

Received 12 December 1980

An analytic solution is found for SIT with two co-propagating pulses which are doubly resonant with two transitions sharing a common excited state. The pulse shapes are the usual hyperbolic secants, but the areas are not $2\pi$. This solution satisfies the atomic equations exactly for all detunings and arbitrary relative amplitudes for the two pulses. The field equations, however, place extra restrictions on the detunings for perfect dispersionless propagation.

Many of the coherence and resonance phenomena which have been studied so extensively with two-level systems interacting with a single quasi-monochromatic field can be generalized to multi-level systems and multiple quasi-monochromatic fields. Theoretical attacks at these problems usually must resort to numerical methods to obtain results. However, every so often analytic solutions are found. In this paper we present such a solution to a generalization of self-induced transparency (SIT) in a three-level system with two applied fields.

There have been several previous treatments of multi-level SIT [1–8] however none of these have noted that in the particular system we will study there is a simple analytic solution with some rather interesting properties. The system we consider is one in which there is a ground state and a low-lying metastable state each coupled to a common upper level by an electric dipole transition. There are two applied field pulses present, one in near resonance with each transition. The two resonance frequencies are assumed to differ enough so that each field can be taken to be resonant with only one of the transitions. We will find an analytic solution for the steady-state propagation of these two pulses traveling together in a medium composed of three-level systems of the type mentioned above. The choice of this "lambda" system and two co-propagating pulses allows any Doppler shifts the pulses may experience to approximately cancel, permitting us to include inhomogeneous broadening. Also, this atomic system has an interesting feature which was pointed out by Gray, Whitley, and Stroud [9] and will be seen to play an important role in this work. Part of the population can become trapped in a coherent superposition of the lower states which is completely decoupled from the applied field.

In the interaction picture, the state vector for a three-level atomic system is

$$|\psi(t,z)\rangle = a(t,z)\exp(-\mathrm{i}\omega_a t)|\mathrm{a}\rangle + b(t,z)\exp(-\mathrm{i}\omega_b t)|\mathrm{b}\rangle + c(t,z)\exp(-\mathrm{i}\omega_c t)|\mathrm{c}\rangle \tag{1}$$

where $z$ is the position of the atom in the material being studied. Taking the atomic system to be of the "lambda" type, we will let states $|\mathrm{a}\rangle$ and $|\mathrm{c}\rangle$ be the closely spaced ground states and $|\mathrm{b}\rangle$ the excited state. As mentioned before, the Doppler shifts experienced by two resonant, monochromatic laser pulses incident on such a system from the same direction will approximately cancel, allowing them to be tuned to within $\Delta$ of the two-photon resonance. Our incident electric field will be taken to be

$$E(t,z) = \boldsymbol{e}_1 \mathcal{E}_1(t,z)\, 2\cos(\omega_1 t - K_1 z) + \boldsymbol{e}_2 \mathcal{E}_2(t,z)\, 2\cos(\omega_2 t - K_2 z), \tag{2}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2$ are unit polarization vectors, $\mathcal{E}_1$, $\mathcal{E}_2$ are the pulse envelopes, $\omega_1$, $\omega_2$ are the pulse frequencies such that $\Delta = \omega_b - \omega_a - \omega_1 = \omega_b - \omega_c - \omega_2$, and $K_1, K_2$ are the wave vector magnitudes.

If we make the following natural restriction for a preserved, undamped pair of pulses traveling together, we can greatly simplify the amplitude equations. We let the pulses have identically shaped envelopes by taking

$$(2\mu_1/\hbar)\ \mathcal{E}_1(t) \equiv \alpha \mathcal{E}(t), \tag{3a}$$

$$(2\mu_2/\hbar)\ \mathcal{E}_2(t) \equiv \beta \mathcal{E}(t), \tag{3b}$$

where $\mu_1 = \langle a|\hat{d}\cdot e_1|b\rangle$, $\mu_2 = \langle c|\hat{d}\cdot e|b\rangle$ are the effective dipole matrix elements and are taken to be real, $\mathcal{E}(t)$ is a normalized pulse shape with peak height one, and $\alpha$, $\beta$ are the Rabi frequencies of the two transitions at the peaks of the pulses. Now, as discovered in ref. [9], if we introduce the two new variables $r$ and $s$ as

$$r(t) = \cos\phi\, a(t) + \sin\phi\, c(t), \tag{4a}$$

$$s(t) = -\sin\phi\, a(t) + \cos\phi\, c(t), \tag{4b}$$

where the angle $\phi$ is defined by

$$\tan\phi \equiv \beta/\alpha \tag{5}$$

we find that Schrödinger's equation leads to

$$\dot{r}(t) = \tfrac{1}{2}\mathrm{i}R\mathcal{E}(t)\exp(-\mathrm{i}\Delta t)\, b(t), \tag{6a}$$

$$\dot{b}(t) = \tfrac{1}{2}\mathrm{i}R\mathcal{E}(t)\exp(\mathrm{i}\Delta t)\, r(t), \tag{6b}$$

$$\dot{s}(t) = 0, \tag{6c}$$

where we have defined a generalized Rabi frequency $R \equiv (\alpha^2 + \beta^2)^{1/2}$, dropped terms oscillating at frequencies greater than $\omega_c - \omega_a$, and suppressed the $z$-dependence. Note that the particular linear combination of states defined by $s(t)$ is not coupled to the excited state by the applied fields and remains undisturbed throughout the interaction. This leaves the equations for $r(t)$ and $b(t)$ in a form identical to those of a two-level atom interacting with a single pulse $\mathcal{E}(t)$ with Rabi frequency $R$. This problem has been solved by McCall and Hahn [10], so we can immediately write down the solution for the pulse envelope:

$$R\mathcal{E}(t) = \frac{2}{\tau}\operatorname{sech}\frac{t - t_0}{\tau}, \tag{7}$$

where, by convention, $\tau$ is the pulse length. So, the generalized pulse envelope is just the hyperbolic secant of the two-level case and the generalized area of $R\mathcal{E}(t)$ is $2\pi$. What does this imply about each individual pulse? The definitions (3a) and (3b) easily show that each individual pulse is a hyperbolic secant with the same pulse length and the individual areas are

$$A_1 = (2\mu_1/\hbar)\int_{-\infty}^{\infty} \mathcal{E}_1(t)\,\mathrm{d}t = \frac{\alpha}{R}2\pi, \tag{8a}$$

$$A_2 = (2\mu_2/\hbar)\int_{-\infty}^{\infty} \mathcal{E}_2(t)\,\mathrm{d}t = \frac{\beta}{R}2\pi. \tag{8b}$$

So, we need $(A_1^2 + A_2^2)^{1/2} = 2\pi$ rather than each individual pulse area being $2\pi$ to produce SIT.

Now we must show the consistency of this solution with Maxwell's equations. We will reinsert the $z$-dependence and use the one-dimensional wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(t,z) = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}P(t,z). \tag{9}$$

The polarization is given by

$$P(t,z) = e_1 P_1(t,z) + e_2 P_2(t,z), \tag{10a}$$

$$P_j(t,z) = N\mu_j \int [u_j(t,z,\Delta)\cos(\omega_j t - K_j z) - v_j(t,z,\Delta)\sin(\omega_j t - K_j z)]\, g_j(\Delta)\,\mathrm{d}\Delta, \tag{10b}$$

where $N$ is the atomic density of the material and $g_j(\Delta)$ is the inhomogeneous lineshape for the $j$th transition. The $u_j$ and $v_j$ are atomic variables defined in the usual manner:

$$u_j = x_j b^* \exp(\mathrm{i}\Delta t) + x_j^* b \exp(-\mathrm{i}\Delta t) \tag{11a}$$

$$v_j = -\mathrm{i}[x_j b^* \exp(\mathrm{i}\Delta t) - x_j^* b \exp(-\mathrm{i}\Delta t)] \tag{11b}$$

$$x_1 = a, \quad x_2 = c, \quad x_3 = r. \tag{11c}$$

To ensure that the pulses keep a constant shape as they travel through the material, we require that $\mathcal{E}_j(t,z) = \mathcal{E}_j(\xi_j)$ where $\xi_j \equiv t - z/V_j$, the "local" time ($V_j$ is the speed of the pulse in the material). For a stationary solution we must have $V_1 = V_2 \equiv V$ so that $\xi_1 = \xi_2 \equiv \xi$. If we assume that the off-resonance dipole moments for each transition respond to each pulse in the same way as the resonant dipole moments except for a possible detuning-dependent reduction in amplitude, we can make the factorization ansatz first utilized by McCall and Hahn:

$$v_j(\xi, \Delta) = F(\Delta)\, v_j(\xi, 0). \tag{12}$$

If the initial state is an incoherent mixture of the lower levels |a⟩ and |c⟩, then we should average all our equations over the random initial phase difference between the complex amplitudes $a(0)$ and $c(0)$. Then, if we also have the initial lower state populations equal, we find that

$$[v_1(\xi, 0)]_{\text{ave.}} = \cos\phi\, v_3(\xi, 0), \tag{13a}$$

$$[v_2(\xi, 0)]_{\text{ave.}} = \sin\phi\, v_3(\xi, 0). \tag{13b}$$

Now, by equating the coefficients of the terms in eq. (9) oscillating at the two different frequencies separately to zero, we find the in-quadrature Maxwell equations

$$(\hbar\alpha/2\mu_1)\dot{\mathcal{E}}(\xi) = \omega_1\mu_1\cos\phi \times \frac{-\pi N v_3(\xi, 0)}{c^2(1/V^2 - 1/c^2)} \int F(\Delta)\, g_1(\Delta)\, \mathrm{d}\Delta \tag{14a}$$

$$(\hbar\beta/2\mu_2)\dot{\mathcal{E}}(\xi) = \omega_2\mu_2\sin\phi \times \frac{-\pi N v_3(\xi, 0)}{c^2(1/V^2 - 1/c^2)} \int F(\Delta)\, g_2(\Delta)\, \mathrm{d}\Delta. \tag{14b}$$

In the limit of homogeneous broadening, $g_j(\Delta)$ is a delta-function. Then with the definition of $\phi$ in eq. (5), eqs. (14) are consistent only if

$$\omega_1\mu_1^2 = \omega_2\mu_2^2, \tag{15}$$

which further implies the following:

$$\frac{\mathcal{E}_2^2/\omega_2}{\mathcal{E}_1^2/\omega_1} = \frac{\beta^2}{\alpha^2}, \tag{16}$$

i.e. the ratio of the photon flux in each beam equals the ratio of the square of the Rabi frequencies. Requirement (15) can be met by adjusting the detuning of the applied pulses from the two-photon resonance.

In the limit of inhomogeneous broadening, $g_j(\Delta)$ is a gaussian with half width $1/T_{2j}^*$ ($T_{2j}^*$ = dipole dephasing time for $j$th transition). When solving the atomic Bloch equations, $F(\Delta)$ was found to be a lorentzian with half width $1/\tau$ ($\tau$ = pulse length). In normal SIT we prefer to have $\tau > T_2^*$ so that any pulse reshaping will occur quickly. Then, $F(\Delta)$ acts as a delta-function in the integrals, picking out the peaks of the lineshape functions $g_j(\Delta)$. For Doppler broadening, these peaks are proportional to $1/\omega_j$. Eqs. (14) are then consistent only if $\mu_1^2 = \mu_2^2$.

If only one pulse is incident on the medium, the solution goes into the normal two-level SIT. If the two lower levels are degenerate (i.e. $\omega_{ca} = 0$), the formalism can be used as shown with the exception that now the generalized pulse $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$. Condition (15) then says $\mu_1^2 = \mu_2^2$ for any detuning. It is also found that $A_1 + A_2 = 2\pi/\sqrt{2}$, i.e. the sum of the pulse areas can be less than $2\pi$.

SIT on degenerate Na transitions has been studied experimentally by Salamo, Gibbs, and Churchill [11]. They observed nearly ideal SIT of a single laser pulse when the dipole moments of the degenerate transitions were equal or even if only the dominant dipole moments were approximately equal. This confirms our result when $\omega_{ca} = 0$. It should also be possible to find appropriate transitions that satisfy conditions (15) in the Na hyperfine structure to test our non-degenerate result.

Very recently, numerical calculations have been performed by Konopnicki for two pulses incident on three-level systems [12]. His results for the "lambda" configuration agree well with the analytic result presented above.

## References

[1] E.M. Belenov and I.A. Poluektov, Sov. Phys. JETP 29 (1969) 754.
[2] M. Takatsuji, Phys. Rev. A4 (1971) 808.
[3] M. Takatsuji, Phys. Rev. A11 (1975) 619.
[4] N. Tan-no, K. Yokoto and H. Inaba, J. Phys. B8 (1975) 339.
[5] N. Tan-no, K. Yokoto and H. Inaba, J. Phys. B8 (1975) 349.
[6] J. Higginbotham, R.T. Deck and D.G. Ellis, Phys. Rev. A16 (1977) 2089.
[7] T.M. Makhiladze, I.G. Sinitsyn and L.A. Shelepin, Sov. J. Quant. Elect. 7 (1977) 739.
[8] M.H. Neyfeh, Phys. Rev. A18 (1978) 2550.
[9] H.R. Gray, R.M. Whitley and C.R. Stroud Jr., Opt. Lett. 3 (1978) 218.
[10] S.L. McCall and E.L. Hahn, Phys. Rev. 183 (1969) 457.
[11] G.J. Salamo, H.M. Gibbs and G.G. Churchill, Phys. Rev. Lett. 33 (1974) 273.
[12] M. Konopnicki, Ph.D. thesis, University of Rochester, 1980.