Comment on ''On the energy levels of a finite square-well potential'' [J. Math. Phys. 41, 4551 (2000)]

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In a recent paper, Paul and Nkemzi[1] presented an elegant expression for the quantized energy levels of the finite square-well potential, derived using the solution of the Riemann–Hilbert boundary problem from the theory of analytic functions. The authors developed an asymptotic expansion for the energy levels $E(p,k)$ in the limit of large p [where $p=\sqrt{2mV_0}L/(2\hbar)$, m is the particle mass, V_0 is the potential depth, L is the length of the finite well, and k is the quantum number], and showed that the finite-well energy levels are approximately equal to those of an infinitely deep square well of length $L'>L$. In this comment, we correct an error in this asymptotic expansion [specifically in Ref. 1, Eq. (28)] and point out that the connection between a finite well and a longer infinite well has been noted previously by other researchers.

Paul and Nkemzi calculated [in Ref. 1, Eqs. (25) and (27)] that in the limit of large p, the quantized values of $z=i\sqrt{(V_0/E)-1}$ take on the asymptotic form

$$z_{p,k}=\pi ik(-1)^k\left(\frac{1}{4p}-\frac{1}{3p^2}-\frac{2(p+1)}{\pi^2k^2}\right)+O\left(\frac{1}{p^3}\right). \tag{1}$$

The energy levels of the finite square well are given in terms of $z_{p,k}$ by

$$E(p,k)=\frac{V_0}{(1-z_{p,k}^2)}=\frac{2\hbar^2p^2}{mL^2(1-z_{p,k}^2)}\equiv\frac{1}{C(p,k)}\,\frac{\pi^2\hbar^2k^2}{2mL^2}, \tag{2}$$

where in the last step we define a correction function $C(p,k)$ that reconciles the finite square-well energy levels with those of an infinitely deep well; that is, $C(p,k)\to 1$ for all k as $p\to\infty$. From Eqs. (1) and (2) we find that

$$C(p,k)=(1-z_{p,k}^2)\,\frac{\pi^2k^2}{4p^2}=1+\frac{2}{p}+\frac{1}{p^2}+O\left(\frac{1}{p^3}\right), \tag{3}$$

where the term $(2/p)$ differs from the results of Paul and Nkemzi. Combining Eqs. (2) and (3), the finite square-well energy levels can be written as

$$E(p,k)=\frac{1}{(1+2p^{-1}+p^{-2})}\,\frac{\pi^2\hbar^2k^2}{2mL^2}+O\left(\frac{1}{p^3}\right)=\frac{p^2}{(p+1)^2}\,\frac{\pi^2\hbar^2k^2}{2mL^2}+O\left(\frac{1}{p^3}\right), \tag{4}$$

which are equivalent to the levels of an infinite square well of length $L'=L(1+1/p)$.

The asymptotic limit of the energy spectrum, Eq. (4), and its interpretation in terms of the infinite square-well spectrum were addressed in detail by Barker *et al.*[2] Calculations of the ''effective length'' or ''tunneling length'' of the finite square-well energy eigenstates beyond the well boundaries at $x=\pm L/2$ were given by Garrett[3] and Rokhsar.[4] To our knowledge, the work by Paul and Nkemzi is the first to derive the finite-well energy levels in an asymptotic expansion in powers of $1/p$; however, power series expansions in the quantum number k have been developed by

Cantrell,[5] Barker *et al.*,[2] and Sprung, Wu, and Martorell.[6] The asymptotic limit also has implications for the behavior of wavepacket states excited in finite potential wells that have been explored by Aronstein and Stroud[7] and by Venugopalan and Agarwal.[8]

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